Curvature forms and characteristic classes

Thomas Maienschein

September 11, 2011
Grassmannians

Definition

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- $G_n(\mathbb{R}^{n+k}) \simeq G_k(\mathbb{R}^{n+k})$ by taking orthogonal complements.
- $G_1(\mathbb{R}^{n+1}) \simeq \mathbb{RP}^n$. 
The **Grassmannian** \( G_n(\mathbb{R}^{n+k}) \) is the set of \( n \)-dimensional subspaces of \( \mathbb{R}^{n+k} \). It is a compact manifold of dimension \( nk \).

- \( G_n(\mathbb{R}^{n+k}) \cong G_k(\mathbb{R}^{n+k}) \) by taking orthogonal complements.
- \( G_1(\mathbb{R}^{n+1}) \cong \mathbb{R}P^n \).

The infinite Grassmannian \( G_n(\mathbb{R}^{\infty}) \) is the direct limit of the sequence:

\[
G_n(\mathbb{R}^n) \subset G_n(\mathbb{R}^{n+1}) \subset \cdots \subset G_n(\mathbb{R}^{n+k}) \subset \cdots
\]

(Direct limit: Take \( \bigcup_{k \geq 0} G_n(\mathbb{R}^{n+k}) \) and choose the finest topology such that every inclusion is continuous)
Grassmannians
The tautological bundle

We can form an \( n \)-plane bundle over \( G_n(\mathbb{R}^{n+k}) \):
Take \( X \in G_n(\mathbb{R}^{n+k}) \). The fiber over \( X \) is \( X \) itself.
So the total space consists of pairs \((n\text{-plane in } \mathbb{R}^{n+k}, \text{vector in that plane})\).
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Definition

The **tautological bundle** over \( G_n(\mathbb{R}^{n+k}) \) is denoted \( \gamma^n(\mathbb{R}^{n+k}) \).

We also have the tautological bundle \( \gamma^n \) over \( G_n(\mathbb{R}^\infty) \).
Grassmannians

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We also have the tautological bundle $\gamma^n$ over $G_n(\mathbb{R}^\infty)$.

- $G_1(\mathbb{R}^2) \cong S^1$, and the tautological bundle “is” the Möbius strip.
Grassmannians
Generalized Gauss Map

**Definition**

Let $M^n \subset \mathbb{R}^{n+k}$. The **Gauss map** $g : M \to G_n(\mathbb{R}^{n+k})$ is given by identifying a tangent space with a subspace of $\mathbb{R}^{n+k}$. 

**Theorem**

For any $n$-plane bundle $\xi$ over $M$, $\xi = g^*(\gamma_n)$ for some $g : M \to G_n(\mathbb{R}^\infty)$. 

Bundles over $M$ are isomorphic iff their classifying maps are homotopic. 

Note: $M$ must be paracompact (open covers admit locally finite refinements). This includes manifolds, metric spaces, CW complexes...
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Classifying Spaces
The structure group

Definition

- A **G-atlas** is a local trivialization of a bundle with transition functions $U \cap V \rightarrow G$.
- A **G-bundle** is a vector bundle with a G-atlas.
- $G$ is called the **structure group**.
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Let $M$ be $n$-dimensional.

- $TM$ is a $GL(n, \mathbb{R})$-bundle.
- If $M$ has a metric, $TM$ is an $O(n)$-bundle.
- If $M$ is also orientable, $TM$ is an $SO(n)$-bundle.
- If $M$ has an almost complex structure, $TM$ is a $U(n)$-bundle.
Define $b_G$ from $\widehat{C\mathcal{W}}^{op} \to \text{Set}$ by $M \mapsto \{G\text{-bundles over } M\}$.
Classifying Spaces

The functors $b_G$ and $B$

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Any $G \subset GL(n, \mathbb{R})$ is representable (Brown representability). So there is always a space $BG$ for which $b_G = [\_ , BG]$. $BG$ is the **classifying space** (Milnor construction).
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$BG$ is the **classifying space** (Milnor construction).

- $BO(n) = G_n(\mathbb{R}^\infty)$.
- $BSO(n) = \tilde{G}_n(\mathbb{R}^\infty)$ (Grassmannian of oriented planes).
- $BU(n) = G_n(\mathbb{C}^\infty)$. 
Characteristic Classes

Cohomology of $BG$

Now we can study of the twisting of a bundle using the classifying map. To study the map indirectly, pull back the cohomology of $BG$.

$H^*(BO(n), \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \ldots, w_n].$

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$H^*(BU(n), \mathbb{R}) \cong \mathbb{R}[c_1, \ldots, c_n]$ (or $\mathbb{Z}$).

For a ring $R$ with $1/2$ (e.g., $R = \mathbb{R}$),

$H^*(BSO(2n+1), R) = R[p_1, \ldots, p_n].$

$H^*(BSO(2n), R) = R[p_1, \ldots, p_{n-1}, e]/(e^2 = p_n/2).$

We call $w_i$ Stiefel-Whitney classes, $c_i$ Chern classes, $p_i$ Pontryagin classes, and $e$ the Euler class.
Now we can study of the twisting of a bundle using the classifying map. To study the map indirectly, pull back the cohomology of $BG$.

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Characteristic Classes

Definition

Let $h^*$ be a cohomology functor (for example, $H^k(\_ \_ \_ , \mathbb{R})$).

A characteristic class is an assignment $c$ of a class in $h^*(M)$ given a $G$-bundle over $M$. It is natural: For $\xi$ over $N$ and $f: M \to N$, $c(f^* \xi) = f^* c(\xi)$.

More succinctly, $c$ is a natural transformation between $bG$ and $h^*$.

Note that pulling back a cohomology class of $BG$ satisfies this definition. In fact, all characteristic classes must be of this form:

Theorem

$Nat(bG, h^*) \cong h^*(BG)$. This follows from contravariant Yoneda's lemma: $bG$ is representable, and $h^*$ can be regarded as a functor to $\text{Set}$.
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$\text{Nat}(b_G, h^*) \simeq h^*(BG)$. 
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Characteristic Classes

Examples of results

A sampling of results:

- $M$ is orientable iff $w_1(M) = 0$.
- $M$ is the boundary of a compact manifold iff $w_i = 0 \ \forall i$.
- If $\mathbb{R}P^{2r}$ is immersed in $\mathbb{R}^{2r+k}$, then $k \geq 2^r - 1$.
- If $M$ has a $q$-frame, then $w_n = \cdots = w_{n-q+1} = 0$.
- Oriented 3-manifolds are parallelizable.
Let $\xi$ be a vector bundle on $M$.  
Let $\Omega^k(M)$ be $k$-forms on $M$.  
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If $\omega \in \Omega^k(M)$ and $s \in \Omega^0(\xi)$, what is $\omega \otimes s$?
We “plug in” $k$ vectors from $TM$, and are just left with $s$.
So it is a $\xi$-valued $k$-form.
Curvature forms
Vector-valued forms

Let \( \xi \) be a vector bundle on \( M \).
Let \( \Omega^k(M) \) be \( k \)-forms on \( M \).
Let \( \Omega^0(\xi) \) be sections of \( \xi \).

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We “plug in” \( k \) vectors from \( TM \), and are just left with \( s \).
So it is a \( \xi \)-valued \( k \)-form.

Definition

Define \( \xi \)-valued \( k \)-forms by \( \Omega^k(\xi) = \Omega^k(M) \otimes_{\Omega^0(M)} \Omega^0(\xi) \).
Curvature forms
Reinterpreting what we have done

Put a metric $\langle , \rangle$ and connection $\nabla$ on $M$.
The connection forms are defined by $\nabla_X E_i = \omega^j_i(X)E_j$. 
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The **connection forms** are defined by $\nabla_X E_i = \omega^j_i(X) E_j$.

We can write this as $\nabla E_i = \omega^j_i \otimes E_j \in \Omega^1(TM)$.

Note also that $\nabla(X^i E_i) = dX^i \otimes E_i + X^i \nabla E_i$. 
Curvature forms

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This motivates a definition of a connection on a general vector bundle $\xi$:

**Definition**

A connection on $\xi$ is an $\mathbb{R}$-linear map $\nabla : \Omega^0(\xi) \to \Omega^1(\xi)$, such that $\nabla(fV) = df \otimes V + f \nabla V$. 
Curvature forms

Generalization

Let $\nabla$ be a connection on a bundle $\xi$ over $M$. Choose a local frame $\{E_i\}$ for $\xi$. Define the connection forms by $\nabla E_i = \omega^j_i \otimes E_j$. Let $A = (\omega^j_i)$. Let $\Omega = dA - A \wedge A$, the matrix of curvature forms. (When $\xi = TM$, everything is the same as before.) $\nabla$ is compatible with a metric $\langle \cdot, \cdot \rangle$ on $\xi$ if:

$$d\langle V, W \rangle = \langle \nabla V, W \rangle + \langle V, \nabla W \rangle$$

Remarks: If $\nabla$ compatible, then $A, \Omega$ antisymmetric WRT orthonormal frames. There is a natural way to pull back connections.
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- There is a natural way to pull back connections.
Invariant polynomials

We want to apply a polynomial $P$ to $\Omega$ and get a globally defined form. Computing $\Omega$ under a change of coordinates gives $T\Omega T^{-1}$. So we want $P(\Omega) = P(T\Omega T^{-1})$. 

Example: Polynomials $\sigma_k$ given by $\det(I + tA) = \sum \sigma_k(A) t^k$. So $\sigma_k(\Omega)$ is a globally defined form on $M$. 
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The Pfaffian

There is a unique polynomial $\text{Pf}$ in the entries of $2n \times 2n$ skew-symmetric matrices such that $\text{Pf}(A)^2 = \det(A)$. 

One can show $\text{Pf}(BAB^T) = \text{Pf}(A) \det(B)$. For $B$ orthogonal, this is invariant. So for an orientable $2n$-plane bundle, $\text{Pf}(\Omega)$ is a globally defined $2n$-form.
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- $\text{Pf}\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a$
- $\text{Pf}\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = af - be + dc$
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- \( \text{Pf} \left( \begin{array}{cc} 0 & a \\ -a & 0 \end{array} \right) = a \)
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- $P(\Omega)$ is globally defined.
- $dP(\Omega) = 0$, so we can consider the cohomology class.
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If $\xi$ is a bundle on $N$ with connection $\nabla$, and $f : M \to N$, then $f^*(P(\Omega_\nabla)) = P(\Omega_{f^*\nabla})$.

So it is natural.
Invariant polynomials

Characteristic classes

One can show that:

- For a bundle with structure group $SO(2n)$, $\text{Pf} \left( \frac{\Omega}{2\pi} \right) = e$.

- For a complex bundle, $\det \left( I + \frac{t\Omega}{2\pi i} \right) = \sum c_k t^k$. 
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- For a complex bundle, $\det \left( I + \frac{t\Omega}{2\pi i} \right) = \sum c_k t^k$.

For a surface, $\Omega = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$

So $\text{Pf} \left( \frac{\Omega}{2\pi} \right) = \frac{\omega}{2\pi} = \frac{K}{2\pi} dV$. 
Chern-Gauss-Bonnet

Integrating characteristic classes (or their cup products, to get the dimension right) over $M$ yields invariant quantities called characteristic numbers.

The Euler class gives the Euler characteristic:

$$\int_M e = \chi(M).$$

(Very rough) sketch of proof ($M$ compact):

Let $\pi: E \to M$ be a $k$-plane bundle. The Thom isomorphism theorem says $\Phi : H^i(M) \cong H^{i+k}(T(E))$. $\Phi(x) = (\pi^* x) \wedge U$, where $U$ is the Thom class.

If $s: M \to E$ is a section, $s^* U$ is the Euler class $e$.

Choose $s$ for which Poincaré-Hopf applies. Then show $\int_M s^* U$ is the sum of indices of zeros of $s$, which is $\chi(M)$. 
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Thomas Maienschein
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**Theorem**

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**Theorem**

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The end!
For details see Spivak I.11, V.13 and Milnor-Stasheff