Clifford Algebras, Division Algebras, and Vector Fields on Spheres

Thomas Maienschein

October 26, 2011
Introduction

In this talk we will:

- Review the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$,
- Define Clifford algebras
- Classify all the Clifford modules of a certain type

Then use our knowledge about Clifford modules to yield results about:

- Vector fields on spheres
- Division algebras
- A few other things

Along the way we will see some “shadows” of Bott periodicity.
Division Algebras

Definition

A finite-dimensional normed division algebra over $\mathbb{R}$ is:

A finite dimensional normed $\mathbb{R}$-vector space with a (not-necessarily associative) $\mathbb{R}$-bilinear product, such that

- There are no zero divisors ($xy = 0 \Rightarrow ||x|| ||y|| = 0$)
- For $0 \neq x \in K$, the left-multiplication map $L_x$ is a linear isomorphism. So there exists $y \in K$ such that $L_x y = 1$. 

(Thomas Maienschein)
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- For $0 \neq x \in \mathbb{K}$, the left-multiplication map $L_x$ is a linear isomorphism.
- So there exists $y \in \mathbb{K}$ such that $L_x y = 1$. 
The first example (other than $\mathbb{R}$!) is $\mathbb{C}$. 
The first example (other than \( \mathbb{R}! \)) is \( \mathbb{C} \).

- As an \( \mathbb{R} \)-vector space, \( \mathbb{C} \approx \mathbb{R}^2 \).
- The absolute value on \( \mathbb{C} \) is the usual norm on \( \mathbb{R}^2 \).
We can construct \( \mathbb{C} \) by the Cayley-Dickson construction: \( \mathbb{C} \) consists of pairs of real numbers with multiplication

\[
(a, b)(c, d) = (ac - d^* b, da + bc^*)
\]

where \( ^* \) is the conjugation on \( \mathbb{R} \) (i.e., it doesn't do anything).
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From this construction, define a conjugation $(a, b)^* = (a^*, -b)$. This is just the usual $(a + bi)^* = a - bi$. 
\( \mathbb{C} \) can be used to describe rotations in \( \mathbb{R}^2 \):
Division Algebras

Complex Numbers

\( \mathbb{C} \) can be used to describe rotations in \( \mathbb{R}^2 \):

\( \mathbb{C} \) acts on \( \mathbb{C} \) by left multiplication.
Identifying \( \mathbb{C} \) with \( \mathbb{R}^2 \) gives a homomorphism \( \mathbb{C} \to M_2(\mathbb{R}) : z \mapsto L_z \).
For \( z \) with \( \|z\| = 1 \), we have \( L_z \in O(2) \).
Division Algebras

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Consider an odd-dimensional sphere \( S^{2n-1} \subset \mathbb{R}^{2n} \).
By identifying \( \mathbb{R}^{2n} \) with \( \mathbb{R}^2 \oplus \cdots \oplus \mathbb{R}^2 \), we can construct a nonvanishing vector field:
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we can construct a nonvanishing vector field:

At the point \(((x_1, y_1), \ldots, (x_n, y_n))\), put the vector
\((i(x_1, y_1), \ldots, i(x_n, y_n)) = ((-y_1, x_1), \ldots, (-y_n x_n))\).
Hamilton knew \( \mathbb{C} \) could be viewed as pairs of real numbers. He tried to define a good multiplication on triples in the 1830s. He became obsessed with this until he defined the quaternions (\( \mathbb{H} \)) in 1842.
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Around 1830 Legendre pointed out that there are no 3-square identities (and therefore no normed division algebra structure on $\mathbb{R}^3$) by showing that $63 = (1^2 + 1^2 + 1^2)(4^2 + 2^2 + 1^2)$ is not a sum of three squares. Was Hamilton aware of this?
Division Algebras

Quaternions

One way to define the quaternions is as follows: \( \mathbb{H} \) consists of 4-tuples of real numbers \( a + bi + cj + dk \), satisfying

\[
    i^2 = j^2 = k^2 = ijk = -1
\]

It follows that \( ij = k, \ jk = i, \ ki = j \), and \( i, j, k \) anticommute.
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It follows that \(ij = k, jk = i, ki = j\), and \(i, j, k\) anticommute.

If \(z = a + bi\) and \(w = c + di\), we have

\[z +wj = (a + bi) + (c + di)j = a + bi + cj + dk\]

So \(\mathbb{H}\) can also be considered as pairs of complex numbers.
We can also use the Cayley-Dickson construction to define $\mathbb{H}$: $\mathbb{H}$ consists of pairs of complex numbers with multiplication

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where $*$ is the conjugation on $\mathbb{C}$.​
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Define a conjugation on $\mathbb{H}$ by $(u, v)^* = (u^*, -v)$. This is just $(a + bi + cj + dk)^* = a - bi - cj - dk$. 
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For $q = a + bi + cj + dk$, define $\|q\| = \sqrt{qq^*} = a^2 + b^2 + c^2 + d^2$. So the norm on $\mathbb{H}$ agrees with the norm on $\mathbb{R}^4$. One can check that $\|pq\| = \|p\|\|q\|$.
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\[ uv = u \times v - u \cdot v \]
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\[ uv = u \times v - u \cdot v \]

It follows that

\[ u \cdot v = -\frac{1}{2}(uv + vu) \quad u \times v = \frac{1}{2}(uv - vu) \]
Quaternions give a nice way to produce rotations of $\mathbb{R}^3$: 

Let $A_q$ be the linear transformation $x \mapsto qxq^{-1}$. Then

$$\begin{align*}
(A_q x)(A_q y) &= \left(qxq^{-1}\right)\left(qyq^{-1}\right) = q(x \times y - x \cdot y)q^{-1} = A_q(x \times y) - x \cdot y
\end{align*}$$

But also

$$\begin{align*}
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So $A_q$ preserves the dot product and cross product. So $A_q \in SO(3)$. 

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So $A_q$ preserves the dot product and cross product. So $A_q \in SO(3)$. 
For a unit vector $u$, define $\exp(u\theta) = \cos \theta + u \sin \theta$. 

Remarks:
Note that the unit quaternions look like $S^3$, and form a Lie group. The Lie algebra is just vectors in $\mathbb{R}^3$. The Lie-theoretic exponential map is the one defined above. There is a lot more to be said about all of this...
Division Algebras

Quaternions

For a unit vector $u$, define $\exp(u\theta) = \cos \theta + u \sin \theta$.

It’s not too hard to show that for $q = \exp(u\theta)$, $A_q$ is a rotation around the vector $u$ by an angle of $2\theta$. 
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- Note that the unit quaternions look like $S^3$, and form a Lie group.
- The Lie algebra is just vectors in $\mathbb{R}^3$.
- The Lie-theoretic exponential map is the one defined above.
- There is a lot more to be said about all of this...
Quaternions can also describe rotations of $\mathbb{R}^4$: 

$\mathbb{H}$ acts on $\mathbb{H}$ by left multiplication. Identifying $\mathbb{H}$ with $\mathbb{R}^4$ gives a homomorphism $\mathbb{H} \to M_4(\mathbb{R})$: $q \mapsto L_q$. For $q$ with $||q|| = 1$, we have $L_q \in O(4)$. Consider a sphere $S^{4n}_n \subset \mathbb{R}^{4n}$. By identifying $\mathbb{R}^{4n}$ with $\mathbb{R}^4 \oplus \cdots \oplus \mathbb{R}^4$, we can construct three nonvanishing vector fields: At the point $((a_1, b_1, c_1, d_1), \ldots, (a_n, b_n, c_n, d_n))$, put the vector $((i(a_1, b_1, c_1, d_1), \ldots, i(a_n, b_n, c_n, d_n)) =$ $((-b_1, a_1, -d_1, c_1), \ldots, (-b_n, a_n, -d_n, c_n))$. Similarly for $j$ and $k$. 
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Consider an sphere \( S^{4n-1} \subset \mathbb{R}^{4n} \).

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Division Algebras
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The next division algebra is the octonions, \( \mathbb{O} \).
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The octonions were defined by Graves, a friend of Hamilton's, shortly after Hamilton defined the quaternions. He used them to give an 8-square identity (which had been discovered by Degen in 1818). Cayley independently defined them in 1845, and they are often called Cayley numbers.
The Cayley-Dickson construction can be used to define $\mathbb{O}$: $\mathbb{O}$ consists of pairs of quaternions with multiplication

$$(p, q)(r, s) = (pr - s^* q, sp + qr^*)$$

where $^*$ is the conjugation on $\mathbb{H}$. 

$\mathbb{O}$ is a real 8-dimensional normed division algebra. $\mathbb{O}$ is not associative.
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Remarks:

- $\mathbb{O}$ is a real 8-dimensional normed division algebra.
- $\mathbb{O}$ is not associative.
Octonions can be used to describe rotations of $\mathbb{R}^8$. 

$\mathbb{O}$ acts on $\mathbb{O}$ by left multiplication. Identifying $\mathbb{O}$ with $\mathbb{R}^8$ gives a map $\mathbb{O} \to M_8(\mathbb{R})$: $a \mapsto L_a$. For $a$ with $||a|| = 1$, we have $L_a \in \mathbb{O}(8)$. The map is not a homomorphism: $L_a L_b x \neq (ab)x = L_{ab} x$. 

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Octonions

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Division Algebras

Octonions

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$S$ has a unit and multiplicative inverses, but has zero divisors. So $S$ is not a division algebra, and we can stop applying Cayley-Dickson.
Clifford Algebras

Definition

We have been defining multiplications on $\mathbb{R}^n$. 

Let $C_\ell^k$ be the free algebra on $\mathbb{R}^k$ subject to the relations $v^2 = -||v||^2$.

Fix an orthonormal basis $\{e_i\}$ of $\mathbb{R}^k$. Then $e_i^2 = -1$, and also $-2 = (e_i + e_j)^2 = e_i^2 + e_i e_j + e_j e_i + e_j^2 = -2 + e_i e_j + e_j e_i$.

Therefore $e_i e_j = -e_j e_i$.

So $C_\ell^k$ is generated by $k$ anticommuting "square roots of 1".

These relations are equivalent to the relations $v^2 = -||v||^2$. 

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Clifford Algebras

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Let’s try the following:
Let \( \mathcal{Cl}_k \) be the free algebra on \( \mathbb{R}^k \) subject to the relations \( v^2 = -\|v\|^2 \).

Fix an orthonormal basis \( \{e_i\} \) of \( \mathbb{R}^k \).
Then \( e_i^2 = -1 \), and also

\[
-2 = (e_i + e_j)^2 = e_i^2 + e_i e_j + e_j e_i + e_j^2 = -2 + e_i e_j + e_j e_i
\]

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So \( \mathcal{Cl}_k \) is generated by \( k \) anticommuting “square roots of 1”.
The relations are equivalent to the relations \( v^2 = -\|v\|^2 \).
Clifford Algebras

Definition

These algebras (sometimes referred to as geometric algebras) were defined by Clifford in 1876. We will see that $\mathcal{C} \ell_1 \simeq \mathbb{C}$ and $\mathcal{C} \ell_2 \simeq \mathbb{H}$, and $\mathcal{C} \ell_k$ generalizes those algebras. We can’t expect $\mathcal{C} \ell_3 \simeq \mathbb{O}$, since the former is associative but the latter is not.
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There are useful for studying $\text{Spin}(n)$ and defining spinors. One can also construct Clifford bundles on a manifold, bundles of Clifford modules, spinor bundles, and other very fancy things. Lawson and Michelsohn’s Spin Geometry has more about this.
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In this talk we will only be interested in finding out when $\mathbb{R}^n$ admits a $\mathcal{C} \ell_k$-module structure. First we will show some basic properties of $\mathcal{C} \ell_k$ to get a feel for it.
Some general observations:

We can divide by nonzero vectors:

\[ v^2 = -||v||^2, \]

so

\[ v^{-1} = -v/||v||^2. \]

There are zero divisors. For example, in \( \mathbb{C}^3 \):

\[ (e_1 e_3 + e_2)(e_2 e_3 + e_1) = e_1 e_3 e_2 e_3 + e_1 e_3 e_1 + e_2 e_2 e_3 + e_2 e_1 = 0. \]
Clifford Algebras
Basic Properties

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(e_1 e_3 + e_2)(e_2 e_3 + e_1) = e_1 e_3 e_2 e_3 + e_1 e_3 e_1 + e_2^2 e_3 + e_2 e_1
\]

\[
= e_1 e_2 + e_3 - e_3 - e_1 e_2 = 0.
\]
Clifford Algebras

Basic Properties

As a vector space, it is generated by elements like $e_{i_1} e_{i_2} \cdots e_{i_m}$.
Clifford Algebras

Basic Properties

As a vector space, it is generated by elements like $e_{i_1} e_{i_2} \cdots e_{i_m}$. We can rearrange so that $i_1 \leq \cdots \leq i_m$. Unlike $\bigwedge \mathbb{R}^k$, there is no $\mathbb{Z}$-grading. For example, $(e_1)(e_1 e_2) = -e_2$. However, cancellation always occurs in pairs, so there is a $\mathbb{Z}_2$-grading.
Clifford Algebras
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Clifford Algebras

Basic Properties

Let’s see what happens when \( \mathcal{C}_2 \).

Multiply two general vectors:

\[
(u_1 e_1 + u_2 e_2)(v_1 e_1 + v_2 e_2) = u_1 v_1 e_2 e_1 + u_1 v_2 e_1 e_2 + u_2 v_1 e_2 e_1 + u_2 v_2 e_2 e_2 = -u \cdot v + \left| u_1 u_2 v_1 v_2 \right| e_1 e_2.
\]

So, for example:

\[
u \cdot v = -\frac{1}{2}(uv + vu).
\]

In fact, this is true in \( \mathcal{C}_k \).
Let’s see what happens when $\mathcal{C}\ell_2$.

Multiply two general vectors:

$$(u_1 e_1 + u_2 e_2)(v_1 e_1 + v_2 e_2) = u_1 v_1 e_1^2 + u_1 v_2 e_1 e_2 + u_2 v_1 e_2 e_1 + u_2 v_2 e_2^2$$

$$= -u \cdot v + \left| \begin{array}{cc} u_1 & u_2 \\ v_1 & v_2 \end{array} \right| e_1 e_2$$
Let’s see what happens when $\mathcal{C}l_2$.

Multiply two general vectors:

\[(u_1 e_1 + u_2 e_2)(v_1 e_1 + v_2 e_2) = u_1 v_1 e_1^2 + u_1 v_2 e_1 e_2 + u_2 v_1 e_2 e_1 + u_2 v_2 e_2^2 = -u \cdot v + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} e_1 e_2\]

So, for example:

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Multiply two general vectors:

$$(u_1 e_1 + u_2 e_2)(v_1 e_1 + v_2 e_2) = u_1 v_1 e_1^2 + u_1 v_2 e_1 e_2 + u_2 v_1 e_2 e_1 + u_2 v_2 e_2^2$$

$$= -u \cdot v + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} e_1 e_2$$

So, for example:

$$u \cdot v = -\frac{1}{2}(uv + vu)$$

In fact, this is true in $\mathcal{C}\ell_k$. 
Let $V$ be an $\mathbb{R}$-vector space with a quadratic form $Q$. Define $\mathcal{C}\ell(V, Q)$ to be the free algebra on $V$ subject to $v^2 = Q(v)$. (That is, $\bigotimes V / I$, where $I$ is generated by $v \otimes v - Q(v)$.)
Let $V$ be an $\mathbb{R}$-vector space with a quadratic form $Q$. Define $\mathcal{C}\ell(V, Q)$ to be the free algebra on $V$ subject to $v^2 = Q(v)$. (That is, $\bigotimes V/I$, where $I$ is generated by $v \otimes v - Q(v)$)

We were looking at $\mathcal{C}\ell_k = \mathcal{C}\ell(\mathbb{R}^k, -1)$. Another Clifford algebra we will need is $\mathcal{C}\ell'_k = \mathcal{C}\ell(\mathbb{R}^k, I)$. The basic properties of $\mathcal{C}\ell'_k$ are essentially the same as for $\mathcal{C}\ell_k$. 

Clifford Algebras
Universal Property
The algebra $\mathcal{C}(V, Q)$ has the following universal property:

Let $A$ be an associative $R$--algebra. Define an $R$--linear map

$\phi : V \rightarrow A$ such that

$\phi(v)^2 = Q(v)$.

Then there exists a unique $\tilde{\phi} : \mathcal{C}(V, Q) \rightarrow A$ such that

$V \xrightarrow{i} \mathcal{C}(V, Q) \xrightarrow{\tilde{\phi}} A$.

Where $i : V \rightarrow \mathcal{C}(V, Q)$ is the natural inclusion.
Clifford Algebras

Universal Property

The algebra $\mathcal{C}_\ell(V, Q)$ has the following universal property:

Let $A$ be an associative $\mathbb{R}$-algebra.
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Then there exists a unique $\tilde{\phi} : \mathcal{C}\ell(V, Q) \to A$ such that TFDC:

$$
\begin{array}{c}
V \\
\downarrow \phi
\end{array}
\xrightarrow{i}
\begin{array}{c}
\mathcal{C}\ell(V, Q) \\
\downarrow \tilde{\phi}
\end{array}
\xleftarrow{\sim}
\begin{array}{c}
A
\end{array}
$$

Where $i : V \to \mathcal{C}\ell(V, Q)$ is the natural inclusion.
For this talk we want to know when $\mathbb{R}^n$ has the structure of a $\mathcal{C}\ell_k$-module.
Clifford Algebras
Universal Property

For this talk we want to know when $\mathbb{R}^n$ has the structure of a $\mathbb{C}\ell_k$-module.

A homomorphism $\tilde{\phi} : \mathbb{C}\ell_k \to M_n(\mathbb{R})$ yields such a structure.

Such homomorphisms are induced by maps $\phi : \mathbb{R}^k \to M_n(\mathbb{R})$, such that:

- $\phi(e_i)^2 = -I$
- $\phi(e_i)\phi(e_j) = -\phi(e_j)\phi(e_i)$

That is, we have $k$ matrices satisfying these relations.
Clifford Algebras

Universal Property

For this talk we want to know when $\mathbb{R}^n$ has the structure of a $\mathcal{C}\ell_k$-module.

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That is, we have $k$ matrices satisfying these relations.

It turns out that determining the (non-)existence of a $\mathcal{C}\ell_k$-module structure will yield useful results about division algebras, vector fields on spheres, $n$-square identities, cross products, and so on.
Clifford Algebras

Periodicity

Recall that

- \( \mathcal{C}_k \) denotes \( \mathcal{C}(k, -1) \) (i.e., \( v^2 = -\|v\|^2 \)),
- \( \mathcal{C}_k' \) denotes \( \mathcal{C}(k, 1) \) (i.e., \( v^2 = \|v\|^2 \))

Let \( M_n(\mathbb{K}) \) denote \( n \times n \) matrices with entries in \( \mathbb{K} \).
Theorem

- $\mathbb{Cl}_1 = \mathbb{C}$
- $\mathbb{Cl}_2 = \mathbb{H}$
Theorem

- $\mathcal{C}\ell_1 = \mathbb{C}$
- $\mathcal{C}\ell_2 = \mathbb{H}$

Proof.

Map $e_1 \mapsto i$.

Map $e_1 \mapsto i$ and $e_2 \mapsto j$ (and $e_1e_2 \mapsto k$).
Clifford Algebras
Periodicity

**Theorem**

- \( \mathcal{C}_1' = \mathbb{R} \oplus \mathbb{R} \)
- \( \mathcal{C}_2' = M_2(\mathbb{R}) \)
Theorem

- $\mathcal{C}l_1' = \mathbb{R} \oplus \mathbb{R}$
- $\mathcal{C}l_2' = \mathcal{M}_2(\mathbb{R})$

Proof.

Map $e_1 \mapsto (1, -1)$.

Map $e_1 \mapsto \begin{pmatrix} 1 & \varepsilon_1 \\ -1 & \varepsilon_1 \end{pmatrix}$ and $e_2 \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (and $e_1 e_2 \mapsto \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$)
Clifford Algebras

Periodicity

**Theorem**

(2) \( \mathbb{C}^\ell_{k+2} \simeq \mathbb{C}^\ell_k \otimes \mathbb{H} \)

(2') \( \mathbb{C}^\ell'_k \simeq \mathbb{C}^\ell_k \otimes \mathbb{M}_2(\mathbb{R}) \)

**Proof of (2).**

Let \( \{ v_i \} \) be a basis for \( \mathbb{R}^k_{k+2} \), let \( \{ e'_i \} \) be the generators for \( \mathbb{C}^\ell'_k \), and \( \{ e_1, e_2 \} \) for \( \mathbb{C}^\ell_2 \).

Define \( u: \mathbb{R}^k_{k+2} \rightarrow \mathbb{C}^\ell'_k \otimes \mathbb{C}^\ell_2 \) by:

- \( v_i \mapsto e'_i \) for \( i = 1, 2 \),
- \( v_i \mapsto e'_i - 2 \otimes e_1 e_2 \) for \( i > 2 \).

Check \( u(v_i) u(v_j) + u(v_j) u(v_i) = 0 \). This induces \( \tilde{u}: \mathbb{C}^\ell_k \rightarrow \mathbb{C}^\ell'_k \otimes \mathbb{C}^\ell_2 \simeq \mathbb{C}^\ell'_k \otimes \mathbb{H} \).

\( \tilde{u} \) is a bijection (consider it as a map between \( \mathbb{R} \)-vector spaces).
Clifford Algebras

Periodicity

**Theorem**

(2) \( \mathcal{C}_k^{l+2} \simeq \mathcal{C}_k^l \otimes \mathbb{H} \)

(2') \( \mathcal{C}_k^{l+2} \simeq \mathcal{C}_k^l \otimes M_2(\mathbb{R}) \)

**Proof of (2).**

Let \( \{v_i\} \) be a basis for \( \mathbb{R}^{k+2} \),

Let \( \{e'_i\} \) be the generators for \( \mathcal{C}_k^l \), and \( \{e_1, e_2\} \) for \( \mathcal{C}_2 \).
Clifford Algebras

Periodicity

**Theorem**

(2) \( \mathcal{C} \ell_{k+2} \cong \mathcal{C} \ell_k^\prime \otimes \mathbb{H} \)

(2') \( \mathcal{C} \ell_{k+2}^\prime \cong \mathcal{C} \ell_k \otimes M_2(\mathbb{R}) \)

**Proof of (2).**

Let \( \{v_i\} \) be a basis for \( \mathbb{R}^{k+2} \),

Let \( \{e_i^\prime\} \) be the generators for \( \mathcal{C} \ell_k^\prime \), and \( \{e_1, e_2\} \) for \( \mathcal{C} \ell_2 \).

Define \( u : \mathbb{R}^{k+2} \to \mathcal{C} \ell_k^\prime \otimes \mathcal{C} \ell_2 \) by:

- \( v_i \mapsto 1 \otimes e_i \) for \( i = 1, 2 \),

- \( v_i \mapsto e_{i-2}^\prime \otimes e_1 e_2 \) for \( i > 2 \).
Clifford Algebras

Periodicity

**Theorem**

(2) $\mathcal{C}\ell_{k+2} \cong \mathcal{C}\ell'_k \otimes \mathbb{H}$

(2$'$) $\mathcal{C}\ell'_{k+2} \cong \mathcal{C}\ell_k \otimes M_2(\mathbb{R})$

**Proof of (2).**

Let $\{v_i\}$ be a basis for $\mathbb{R}^{k+2}$,

Let $\{e'_i\}$ be the generators for $\mathcal{C}\ell'_k$, and $\{e_1, e_2\}$ for $\mathcal{C}\ell_2$.

Define $u : \mathbb{R}^{k+2} \rightarrow \mathcal{C}\ell'_k \otimes \mathcal{C}\ell_2$ by:

- $v_i \mapsto 1 \otimes e_i$ for $i = 1, 2$,
- $v_i \mapsto e'_{i-2} \otimes e_1 e_2$ for $i > 2$.

Check $u(v_i)^2 = -1$ and $u(v_i)u(v_j) + u(v_j)u(v_i) = 0$.

This induces $\tilde{u} : \mathcal{C}\ell_{k+2} \rightarrow \mathcal{C}\ell'_k \otimes \mathcal{C}\ell_2 \cong \mathcal{C}\ell'_k \otimes \mathbb{H}$.

$\tilde{u}$ is a bijection (consider it as a map between $\mathbb{R}$-vector spaces).

□
Clifford Algebras

Periodicity

Theorem

(4) \( \mathcal{C}_k + 4 \cong \mathcal{C}_k \otimes M_2(\mathbb{H}) \)

(4') \( \mathcal{C}'_k + 4 \cong \mathcal{C}'_k \otimes M_2(\mathbb{H}) \)
Theorem

(4) $\mathcal{C} \ell_{k+4} \simeq \mathcal{C} \ell_{k} \otimes M_2(\mathbb{H})$

(4') $\mathcal{C} \ell'_{k+4} \simeq \mathcal{C} \ell'_{k} \otimes M_2(\mathbb{H})$

Proof of (4).

Use the previous theorem twice:

\[
\mathcal{C} \ell_{k+4} = \mathcal{C} \ell_{(k+2)+2} \simeq (\mathcal{C} \ell_k \otimes M_2(\mathbb{R}) \otimes \mathbb{H}) \simeq \mathcal{C} \ell_{k+2} \otimes \mathbb{H} \simeq \mathcal{C} \ell_k \otimes M_2(\mathbb{R}) \otimes \mathbb{H} \simeq \mathcal{C} \ell_k \otimes M_2(\mathbb{H})
\]
Clifford Algebras
Periodicity

Theorem

(8) \( \mathcal{C}_{\ell_k+8} \cong \mathcal{C}_{\ell_k} \otimes M_{16}(\mathbb{R}) \)

(8') \( \mathcal{C}_{\ell'_k+8} \cong \mathcal{C}_{\ell'_k} \otimes M_{16}(\mathbb{R}) \)

Proof of (8).
Use the previous theorem twice:

\[ \mathcal{C}_{\ell_k} \cong \mathcal{C}_{\ell_k+4} \otimes M_2(\mathbb{H}) \]

\[ \cong \left( \mathcal{C}_{\ell_k} \otimes M_2(\mathbb{H}) \right) \otimes M_2(\mathbb{H}) \]

\[ \cong \mathcal{C}_{\ell_k} \otimes \left( M_2(\mathbb{H}) \otimes M_2(\mathbb{H}) \right) \]

\[ \cong \mathcal{C}_{\ell_k} \otimes M_{16}(\mathbb{R}) \]

(Because \( H \otimes H \cong M_4(\mathbb{R}) \))
Clifford Algebras

Periodicity

Theorem

\( (8) \quad \mathcal{C}_\ell_{k+8} \simeq \mathcal{C}_\ell_k \otimes M_{16}(\mathbb{R}) \)

\( (8') \quad \mathcal{C}_\ell'_{k+8} \simeq \mathcal{C}_\ell'_k \otimes M_{16}(\mathbb{R}) \)

Proof of (8).

Use the previous theorem twice:

\[
\mathcal{C}_\ell_{k+8} = \mathcal{C}_\ell_{(k+4)+4} \simeq \mathcal{C}_\ell_{k+4} \otimes M_2(\mathbb{H}) \\
\simeq (\mathcal{C}_\ell_k \otimes M_2(\mathbb{H})) \otimes M_2(\mathbb{H}) \\
\simeq \mathcal{C}_\ell_k \otimes (M_2(\mathbb{H}) \otimes M_2(\mathbb{H})) \simeq \mathcal{C}_\ell_k \otimes M_{16}(\mathbb{R})
\]

(Because \( \mathbb{H} \otimes \mathbb{H} \simeq M_4(\mathbb{R}) \))
Finally, we can find all of the $\mathcal{C}\ell_k$ (and all of the $\mathcal{C}\ell'_k$):

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- $\mathcal{C}\ell_3 \simeq \mathcal{C}\ell'_1 \otimes \mathbb{H} \simeq (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H}.$
Finally, we can find all of the $\mathcal{C} \ell_k$ (and all of the $\mathcal{C} \ell'_k$):

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Finally, we can find all of the $\mathcal{C}\ell_k$ (and all of the $\mathcal{C}\ell'_k$):

$$
\begin{array}{|c|c|}
\hline
k & C\ell_k \\
\hline
1 & \mathbb{C} \\
2 & \mathbb{H} \\
3 & \mathbb{H} \oplus \mathbb{H} \\
4 & \mathcal{M}_2(\mathbb{H}) \\
\hline
\end{array}
$$

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  But $\mathbb{C} \otimes \mathbb{H} \simeq \mathcal{M}_2(\mathbb{C})$.
- $\mathcal{C} \ell_6 \simeq \mathcal{C} \ell'_2 \otimes \mathcal{M}_2(\mathbb{H}) \simeq \mathbb{H} \otimes \mathcal{M}_2(\mathbb{H})$
  But $\mathbb{H} \otimes \mathbb{H} \simeq \mathcal{M}_4(\mathbb{R})$. 
Finally, we can find all of the $\mathcal{C}\ell_k$ (and all of the $\mathcal{C}\ell'_k$):

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**Clifford Algebras**

**Periodicity**

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It will turn out to be useful to know when $\mathbb{R}^n$ has a $\mathbb{C}l_k$-module structure.
Clifford Algebras

Clifford Modules

It will turn out to be useful to know when $\mathbb{R}^n$ has a $\mathbb{C}\ell_k$-module structure.

Given our classification, this theorem tells us everything we need:

**Theorem**

If $K$ is a division algebra,

- $M_n(K)$ has a unique simple module $K^n$,
- $M_n(K) \oplus M_n(K)$ has two, inherited from each summand,
- Every other module is a direct sum of these.

(See e.g. Lang’s Algebra, chapter XVII)
Let $n_k$ denote the smallest $n$ for which $\mathbb{R}^n$ is a simple $\mathcal{C}l_k$-module.
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Observe that $n_{k+8} = 16n_k$, and that $n_k$ gets a lot bigger than $k$. 
Vector Fields on Spheres

Examples

Problem

For a given \( n \), what is the maximal \( k \) such that there exist vector fields \( V_1, \ldots, V_k \) on \( S^{n-1} \) which are orthonormal at each point?
Vector Fields on Spheres

Examples

Problem

For a given $n$, what is the maximal $k$ such that there exist vector fields $V_1, \ldots, V_k$ on $S^{n-1}$ which are orthonormal at each point?

We saw earlier that we could use division algebras to construct some vector fields. This won’t get us very far (as we will see later).
Vector Fields on Spheres

Examples

We can take care of half of the spheres right away:

**Theorem (Hairy Ball Theorem)**

*Even-dimensional spheres have no non-vanishing vector fields.*
Vector Fields on Spheres

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**Theorem (Hairy Ball Theorem)**

Even-dimensional spheres have no non-vanishing vector fields.

**Proof.**

Let $v$ be a non-vanishing vector field on $S^n$. Normalize it.
Vector Fields on Spheres

Examples

We can take care of half of the spheres right away:

**Theorem (Hairy Ball Theorem)**

*Even-dimensional spheres have no non-vanishing vector fields.*

**Proof.**

Let \( v \) be a non-vanishing vector field on \( S^n \). Normalize it.

Define \( F_\theta(x) = x \cos \theta + v_x \sin \theta \).

\( F_\theta \) is a homotopy between the identity \( 1_S \) and the antipodal map \( \alpha \).
We can take care of half of the spheres right away:

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Let $\nu$ be a non-vanishing vector field on $S^n$. Normalize it.

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$F_\theta$ is a homotopy between the identity $1_S$ and the antipodal map $\alpha$.

So $\deg \alpha = \deg 1_S = 1$. But for even $n$, $\deg \alpha = -1$, since $\alpha$ is the composition of $n + 1$ reflections.
In general, we can get a \textit{lower bound} by studying Clifford modules:

\begin{theorem}

If $\mathbb{R}^n$ admits the structure of a $\mathbb{C}\ell^k$-module, then we can construct $k$ orthonormal vector fields on $S^{n-1}$.

\end{theorem}

\textbf{Remarks:}

Fixing a basis for $\mathbb{R}^n$, a $\mathbb{C}\ell^k$-module structure on $\mathbb{R}^n$ is a ring homomorphism $\varphi: \mathbb{C}\ell^k \rightarrow M_n(\mathbb{R})$. We can think of this as a choice of matrices $U_i = \varphi(e_i)$ satisfying:

\begin{align*}
U_i^2 &= -I \\
U_i U_j &= -U_j U_i \quad \text{for} \quad i \neq j
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In general, we can get a *lower bound* by studying Clifford modules:

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Vector Fields on Spheres

Application of Clifford Modules

In general, we can get a *lower bound* by studying Clifford modules:

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We can choose a basis for $\mathbb{R}^n$ so that $U_i \in O(n)$ for all $i$: 
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Let $\Gamma$ be the group generated by $\{U_1, \ldots, U_k\}$. 

Computing $\langle U_i x, U_i y \rangle$ just permutes the sum defining $\langle x, y \rangle$.
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$$\langle x, y \rangle = \frac{1}{|\Gamma|} \sum_{U \in \Gamma} (Ux, Uy)$$
Vector Fields on Spheres
Application of Clifford Modules

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Choose an orthonormal basis for $(\, , )$. 
Vector Fields on Spheres
Application of Clifford Modules

For \( x \in S^{n-1} \), we may consider \( x \in \mathbb{R}^n \) with \( \|x\| = 1 \).
The vectors \( \{x, U_1x, \ldots, U_kx\} \) are mutually orthogonal:

Proof Part 1
\[
\langle U_i x, x \rangle = \langle -x, U_i x \rangle \quad \text{(since} \ U_i \in O(n) \text{)}
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\[
= -\langle U_i x, x \rangle \quad \text{(since} \ U_2i = -1)\]

Proof Part 2
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\langle U_i x, U_j x \rangle = \langle -x, U_i U_j x \rangle \quad \text{(since} \ U_i U_j = -U_j U_i)\]
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So we have \( k \) vector fields on \( S^{n-1} \).
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\[
= \langle U_j x, -U_i x \rangle \quad (\text{since } U_j \in O(n) \text{ and } U_j^2 = -1)
\]
\[
= -\langle U_i x, U_j x \rangle
\]

So we have $k$ vector fields on $S^{n-1}$. 
Precisely how many vector fields can we construct in this way?
Precisely how many vector fields can we construct in this way?

In other words, given $n$, what is the largest $k$ such that $\mathbb{R}^n$ admits the structure of a $\mathcal{C}ℓ_k$-module?
Vector Fields on Spheres
Application of Clifford Modules

Recall that $n_k$ is the smallest $n$ for which $\mathbb{R}^n$ is a simple $\mathbb{C}\ell_k$-module.

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$(n_{k+8} = 16n_k)$
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$\rho(n)$ is the largest $k$ such that $n_k | n$.

These are called Radon-Hurwitz numbers.

Examples:

For odd $n$, $\mathbb{R}^n$ is not a $\mathcal{C}\ell_k$-module.

$\mathbb{R}^4$ admits a $\mathcal{C}\ell_3$-module structure.

$\mathbb{R}^8$ also does (since it is $\mathbb{R}^4 \oplus \mathbb{R}^4$).

But $\mathbb{R}^8$ is also a $\mathcal{C}\ell_7$-module.
Recall that $n_k$ is the smallest $n$ for which $\mathbb{R}^n$ is a simple $\mathcal{C}\ell_k$-module.

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($n_{k+8} = 16n_k$)
Vector Fields on Spheres

Application of Clifford Modules

Recall that $n_k$ is the smallest $n$ for which $\mathbb{R}^n$ is a simple $\mathcal{C}_k$-module.

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Write $n = 16^a \ 2^b \ m$,
(where $m$ is odd, $0 \leq b \leq 3$)

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Observe that:

- $n_{8a} = 16^a$,
- $n_{8a+1} = 2 \cdot 16^a$
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So $\rho(n) = 8a + 2^b - 1$. 
Here is a table that shows how many orthonormal VFs we can construct:

<table>
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<tr>
<th># VFs</th>
<th>$S^1$</th>
<th>$S^3$</th>
<th>$S^5$</th>
<th>$S^7$</th>
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<tbody>
<tr>
<td>$S^1$</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>7</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>$S^{17}$</td>
<td>1</td>
<td>3</td>
<td>1</td>
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It turns out that we constructed everything: $\rho(n)$ is the maximal number of linearly independent VFs on $S^{n-1}$. This is much harder (proved by J.F. Adams in 1962 using Adams operations in $K$-theory).
Vector Fields on Spheres

The Result

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Problem

When can we make $\mathbb{R}^n$ into a division algebra?
Division Algebras

Examples

Problem

When can we make $\mathbb{R}^n$ into a division algebra?

Remarks:

- There are various theorems with differing assumptions about whether the algebra is commutative, associative, or normed.
- We will *not* require commutativity or associativity. We *will* require the algebra to be normed.
- Examples include $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ (that’s all of them, in fact).
Let’s see what Clifford modules say about such division algebras:

**Theorem**

Let $K$ be a finite-dimensional normed division algebra over $\mathbb{R}$.

If $\dim K = n (> 1)$, then $\mathbb{R}^n$ admits the structure of a $\mathbb{C}^{\ell n-1}$-module.

We will show that this implies $n = 2, 4, \text{or } 8$.

**Remarks:**

Corollary: $\dim K = n = \Rightarrow S^{n-1}$ is parallelizable.

That isn’t useful unless we know which spheres are parallelizable.

(our lower bound $\rho(n)$ is sharp, but that is hard to prove).

If we drop the normed condition, it still implies $S^{n-1}$ is parallelizable.

It no longer implies $\mathbb{R}^n$ is a $\mathbb{C}^{\ell n-1}$-module.
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Division Algebras
Application of Clifford Modules

Let $K$ be our division algebra, $\dim K = n$.
As an $\mathbb{R}$-vector space, it is $\mathbb{R}^n$, and it has some multiplication.
Division Algebras
Application of Clifford Modules

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We will assume the norm comes from an inner product $\langle , \rangle$ on $\mathbb{R}^n$.
(This isn’t obvious, but it can be proven)
Fix an orthonormal basis for $\langle , \rangle$.

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Let $\text{Im}(\mathbb{K})$ denote the elements of $\mathbb{K}$ orthogonal to 1 ($\mathbb{R}^{n-1}$ as a VS).
Left multiplication is an $\mathbb{R}$-linear map $\mathbb{K} \to \mathbb{K}$. So there is a map

$$\text{Im}(\mathbb{K}) \xrightarrow{\phi} M_n(\mathbb{R}) : v \mapsto \left( x \mapsto L_v x \right)$$

Thomas Maienschein ()
Clifford Algebras, Division Algebras, and Vector Fields on Spheres
October 26, 2011 47 / 1
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$$\text{Im}(K) \xrightarrow{\phi} M_n(\mathbb{R}) : v \mapsto \left( x \xrightarrow{L_v} vx \right)$$

The strategy is to show that for $v \in \text{Im}(K)$ with $\|v\| = 1$, $L_v^2 = -I$. That will induce a map $\tilde{\phi} : C\ell_{n-1} \to M_n(\mathbb{R})$. 

Let $\nu \in \text{Im}(\mathbb{K})$, $\|\nu\| = 1$.
$L_{\nu} \in O(n)$ (since $\|L_{\nu}x\| = \|\nu x\| = \|\nu\|\|x\| = \|x\|$).
Division Algebras
Application of Clifford Modules

Let \( v \in \text{Im}(\mathbb{K}), \|v\| = 1 \).
\( L_v \in O(n) \) (since \( \|L_v x\| = \|v x\| = \|v\| \|x\| = \|x\| \)).

Proof that \( L_v^2 = -I \)

Let \( w = (v + 1)/\sqrt{2} \).
Since \( \|w\| = 1 \), \( L_w = (L_v + I)/\sqrt{2} \in O(n) \).
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**Proof that $L_v^2 = -I$**

Let $w = (v + 1)/\sqrt{2}$.

Since $\|w\| = 1$, $L_w = (L_v + I)/\sqrt{2} \in O(n)$.

Therefore:

\[
I = L_w L_w^* = \frac{1}{2} (L_v + I)(L_v^* + I)
\]

\[
= \frac{1}{2} (L_v L_v^* + L_v + L_v^* + I) = I + \frac{1}{2} (L_v + L_v^*)
\]

So $L_v^2 = (-L_v^*)(L_v) = -I$.  

---

**Division Algebras**

Application of Clifford Modules

---

**Clifford Algebras, Division Algebras, and Vector Fields on Spheres**

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For $\mathbb{R}^n$ to be a $\mathcal{C}\ell_{n-1}$-module, we need $n_{n-1} \mid n$. 
Division Algebras

The Result

For \( \mathbb{R}^n \) to be a \( \mathcal{C}l_{n-1} \)-module, we need \( n_{n-1} \mid n \).

<table>
<thead>
<tr>
<th>( n )</th>
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<th>( \mathcal{C}l_{n-1} )</th>
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<td>2</td>
<td>2</td>
<td>( \mathbb{C} )</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>( \mathbb{H} )</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>( \mathbb{H} \oplus \mathbb{H} )</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>( M_2(\mathbb{H}) )</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
<td>( M_4(\mathbb{C}) )</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>( M_8(\mathbb{R}) )</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>( M_8(\mathbb{R}) \oplus M_8(\mathbb{R}) )</td>
</tr>
<tr>
<td>9</td>
<td>16</td>
<td>( M_{16}(\mathbb{R}) )</td>
</tr>
</tbody>
</table>

This only happens for \( n = 2, 4, \) or \( 8 \). After \( n = 8 \), \( n_{n-1} > n \).
We proved:

**Theorem**

Let $K$ be a finite-dimensional normed division algebra over $\mathbb{R}$. Then $\dim K = 1, 2, 4, \text{ or } 8$. 

Remarks: The same result is true if we drop the normed condition. This is much harder (proved by Kervaire, and, independently, by Bott and Milnor, in 1958). It's not much harder to show $K = \mathbb{R}, \mathbb{C}, \mathbb{H}, \text{ or } \mathbb{O}$ (Hurwitz's theorem). This part is not true if we drop the normed condition.
We proved:

**Theorem**

Let $\mathbb{K}$ be a finite-dimensional normed division algebra over $\mathbb{R}$. Then $\dim \mathbb{K} = 1, 2, 4, \text{ or } 8$.

**Remarks:**

- The same result is true if we drop the normed condition.
- This is much harder (proved by Kervaire, and, independently, by Bott and Milnor, in 1958).
- It’s *not* much harder to show $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \text{ or } \mathbb{O}$ (Hurwitz’s theorem). This part is *not* true if we drop the normed condition.
Applications
Square Identities

Consider this two-square identity:

\[(a_1^2 + a_2^2)(b_1^2 + b_2^2) = c_1^2 + c_2^2\]
\[c_1 = a_1 b_1 - a_2 b_2\]
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There is also Euler’s four-square identity:

\[(a_1^2 + a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2) = c_1^2 + c_2^2 + c_3^2 + c_4^2\]
\[c_1 = a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4\]
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There is also Degen’s eight-square identity (I’m not typing that one!)
Applications

Square Identities

These identities just come from the normed division algebra structures on $\mathbb{R}^n$ for $n = 2, 4, 8$: $\|a\|^2 \|b\|^2 = \|c\|^2$, $c = ab$, where $a, b, c \in \mathbb{C}, \mathbb{H}, \text{or} \mathbb{O}$. 
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Theorem

Define an $n$-square identity to be an expression of the form $(\sum a_i^2) (\sum b_i^2) = \sum c_i^2$, where $c_i$ is bilinear in the $a$’s and $b$’s.

There only exist 1-, 2-, 4-, and 8-square identities.
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These identities just come from the normed division algebra structures on \( \mathbb{R}^n \) for \( n = 2, 4, 8 \): \( \|a\|^2 \|b\|^2 = \|c\|^2 \), \( c = ab \), where \( a, b, c \in \mathbb{C}, \mathbb{H}, \text{ or } \mathbb{O} \).

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\]

*There only exist 1-, 2-, 4-, and 8-square identities.*

Start by defining an \( \mathbb{R} \)-algebra structure on \( \mathbb{R}^n \) by:

\[
(a_1, \ldots, a_n)(b_1, \ldots, b_n) = (c_1, \ldots, c_n)
\]

This algebra preserves the norm on \( \mathbb{R}^n \). It has no zero-divisors.
Applications
Square Identities

We can’t apply our division algebra theorem; there is no apparent unit!

To fix, we perform a “mutation”:
Choose $u \in \mathbb{R}^n$ with $||u|| = 1$.

Define a new product
\[ a \ast b = (R - 1_u a)(L - 1_u b), \]
where $L_u$ and $R_u$ are left and right multiplication by $u$.

The new product still gives us an $\mathbb{R}$-algebra preserving $||\cdot||$.

The multiplicative identity is $u^2$.

Let $x \in \mathbb{R}^n$ and $y = L^{-1}u x$. Then:
\[ u^2 \ast x = u^2 \ast L_u y = (R - 1_u u^2)(L - 1_u L_u y) = uy = x \]
(Where $R - 1_u u^2 = u$ since $R_u u^2 = u^2$).

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\]

(Where \( R_u^{-1}u^2 = u \) since \( R_u u = u^2 \)).
This mutation turned $\mathbb{R}^n$ into finite-dimensional normed division algebra over $\mathbb{R}$, so $n = 1, 2, 4, \text{ or } 8$, proving the theorem.
Applications

Cross Products

Cross products exist in $\mathbb{R}^3$ and $\mathbb{R}^7$ (Im($\mathbb{H}$) and Im($\mathbb{O}$), respectively).

Theorem

Suppose there is a cross product on $\mathbb{R}^n$, $n \geq 3$, such that $u \times v$ is bilinear in $u$ and $v$, $u \times v$ is perpendicular to $u$ and $v$, $||u \times v||^2 = ||u||^2 ||v||^2 - (u \cdot v)^2$.

Then $n = 3$ or $7$.

Existence of such a cross product makes $\mathbb{R} \oplus \mathbb{R}^n$ into a normed division algebra by defining:

$$(a, v)(b, w) = (ab - v \cdot w, aw + bv + v \times w)$$

(This from Massey's paper "Cross products of vectors in higher dimensional Euclidean spaces").
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**Theorem**

*Finite-dimensional, normed, real division algebras occur only in dimension 1, 2, 4, 8.*

But in fact we can drop the “normed”. That turns out to be much harder. (Kervaire, Bott, Milnor 1958).
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**Theorem**

*There are at least $\rho(n)$ linearly independent vector fields on $S^{n-1}$.*

But in fact there are also *at most* $\rho(n)$. Also much harder. (Adams 1962).
Bott Periodicity

The “optimal” results shown use $\tilde{K}$-theory.
There is a relationship between $\tilde{KO}$ and Clifford modules, described in the paper “Clifford Modules” of Atiyah, Bott, and Shapiro.

$$
\tilde{KO}(S^1) \cong \mathbb{Z}_2 \\
\tilde{KO}(S^2) \cong \mathbb{Z}_2 \\
\tilde{KO}(S^3) \cong 0 \\
\tilde{KO}(S^4) \cong \mathbb{Z}_2 \\
\tilde{KO}(S^5) \cong 0 \\
\tilde{KO}(S^6) \cong 0 \\
\tilde{KO}(S^7) \cong 0 \\
\tilde{KO}(S^8) \cong \mathbb{Z}_2 \\
\tilde{KO}(S^8+\mathcal{K}) \cong \tilde{KO}(S^k).
$$
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The statement of (one version of) Bott periodicity:

\[ \begin{align*}
\widetilde{KO}(S^1) &\cong \mathbb{Z}_2 \\
\widetilde{KO}(S^2) &\cong \mathbb{Z}_2 \\
\widetilde{KO}(S^3) &\cong 0 \\
\widetilde{KO}(S^4) &\cong \mathbb{Z} \\
\widetilde{KO}(S^5) &\cong 0 \\
\widetilde{KO}(S^6) &\cong 0 \\
\widetilde{KO}(S^7) &\cong 0 \\
\widetilde{KO}(S^8) &\cong \mathbb{Z}
\end{align*} \]

And $\widetilde{KO}(S^{8+k}) \cong \widetilde{KO}(S^k)$. 
Bott Periodicity

There are isomorphisms $L_k \simeq \tilde{KO}(S^k)$,
where $L_k \simeq \text{coker}(N_k \to N_{k-1})$,
where $N_k$ is the free abelian group generated by simple $\mathcal{C}\ell_k$-modules, and
$N_k \to N_{k-1}$ is induced by an inclusion $\mathcal{C}\ell_{k_1} \to \mathcal{C}\ell_k$. 
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