Desingularizing the Boundary of the Moduli Space of Genus One Stable Quotients

Thomas D. Maienschein

June 9, 2014
Background
Moduli Spaces

What is a moduli space?
Example: Smooth plane conics.
Background
Moduli Spaces

**Example:** Smooth plane conics.

A plane conic is a curve in $\mathbb{P}^2$ defined by a degree 2 polynomial:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz = 0$$
Example: Smooth plane conics.

A plane conic is a curve in $\mathbb{P}^2$ defined by a degree 2 polynomial:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz = 0$$

We can regard this as a point $[A : B : C : D : E : F] \in \mathbb{P}^5$. 
Background
Moduli Spaces

Example: Smooth plane conics.

A plane conic is a curve in $\mathbb{P}^2$ defined by a degree 2 polynomial:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz = 0$$

We can regard this as a point $[A : B : C : D : E : F] \in \mathbb{P}^5$.

$\mathbb{P}^5$ is a moduli space of plane conics:

$$\{\text{Points of } \mathbb{P}^5\} \xleftrightarrow{1-1} \{\text{Plane conics}\}$$
Background
Moduli Spaces

What about the moduli space of smooth plane conics?
What about the moduli space of *smooth* plane conics? 

The conic 

\[ Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz = 0 \]

is singular exactly when 

\[ \begin{vmatrix} 2A & D & E \\ D & 2B & F \\ E & F & 2C \end{vmatrix} = 0. \]
Background
Moduli Spaces

What about the moduli space of smooth plane conics?

The conic

\[ Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz = 0 \]

is singular exactly when

\[
\begin{vmatrix}
2A & D & E \\
D & 2B & F \\
E & F & 2C \\
\end{vmatrix} = 0.
\]

This is a degree 3 polynomial in the coordinates of \( \mathbb{P}^5 \).
It defines a hypersurface \( \Delta \subset \mathbb{P}^5 \).
So:

- \( \mathbb{P}^5 \setminus \Delta \) is the moduli space of smooth conics.
So:

- $\mathbb{P}^5 \setminus \Delta$ is the moduli space of smooth conics.
- $\mathbb{P}^5$ is a compactification of the moduli space.

What is the boundary like?

$\Delta$ is itself singular. The singular locus is $\Delta$ double, the locus of “double lines” $\left(\alpha x + \beta y + \gamma z\right)^2 = 0$.

Blowing up along $\Delta$ double will resolve the singularities in the boundary.
So:

- \( \mathbb{P}^5 \setminus \Delta \) is the moduli space of smooth conics.
- \( \mathbb{P}^5 \) is a **compactification** of the moduli space.
- \( \Delta \) is the **boundary** of the compactification.

What is the boundary like?
Background

Moduli Spaces

So:

- $\mathbb{P}^5 \setminus \Delta$ is the moduli space of smooth conics.
- $\mathbb{P}^5$ is a compactification of the moduli space.
- $\Delta$ is the boundary of the compactification.

What is the boundary like?

- $\Delta$ is itself singular.
So:

- $\mathbb{P}^5 \setminus \Delta$ is the moduli space of smooth conics.
- $\mathbb{P}^5$ is a **compactification** of the moduli space.
- $\Delta$ is the **boundary** of the compactification.

What is the boundary like?

- $\Delta$ is itself singular.
- The singular locus is $\Delta_{\text{double}}$, the locus of “double lines”
  
  \[(ax + by + cz)^2 = 0\]
Background
Moduli Spaces

So:

- \( \mathbb{P}^5 \setminus \Delta \) is the moduli space of smooth conics.
- \( \mathbb{P}^5 \) is a **compactification** of the moduli space.
- \( \Delta \) is the **boundary** of the compactification.

What is the boundary like?

- \( \Delta \) is itself singular.
- The singular locus is \( \Delta_{\text{double}} \), the locus of “double lines”

\[
(\alpha x + \beta y + \gamma z)^2 = 0
\]

- Blowing up along \( \Delta_{\text{double}} \) will resolve the singularities in the boundary.
The moduli space of genus one stable quotients is a nonsingular compactification of the moduli space of maps from smooth genus 1 curves into projective space.

(By curve we mean a 1-dimensional complex projective variety)
Maps $C \to \mathbb{P}^{n-1}$ can be specified by a short exact sequence of bundles.
Maps $C \to \mathbb{P}^{n-1}$ can be specified by a short exact sequence of bundles.

Consider

$$0 \to S \to \mathcal{O}_C^\oplus n \to Q \to 0$$

where $S$ is a line bundle.
Maps $C \to \mathbb{P}^{n-1}$ can be specified by a short exact sequence of bundles.

- Consider
  \[
  0 \to S \to \mathcal{O}_C^\oplus n \to Q \to 0
  \]
  where $S$ is a line bundle.
- Dualize the first map to get $n$ sections of $S^\vee$:
  \[
  \mathcal{O}_C^\oplus n \to S^\vee
  \]
Maps and Quot Schemes

Maps $C \to \mathbb{P}^{n-1}$ can be specified by a short exact sequence of bundles.

- Consider

$$0 \to S \to \mathcal{O}_C^\oplus n \to Q \to 0$$

where $S$ is a line bundle.

- Dualize the first map to get $n$ sections of $S^\vee$:

$$\mathcal{O}_C^\oplus n \to S^\vee$$

- Define a map by:

$$p \mapsto [s_1(p) : \cdots : s_n(p)] \in \mathbb{P}^{n-1}$$
Maps $C \rightarrow \mathbb{P}^{n-1}$ can be specified by a short exact sequence of bundles.

- Consider

\[ 0 \rightarrow S \rightarrow \mathcal{O}_C^\oplus n \rightarrow Q \rightarrow 0 \]

where $S$ is a line bundle.

- Dualize the first map to get $n$ sections of $S^\vee$:

\[ \mathcal{O}_C^\oplus n \rightarrow S^\vee \]

- Define a map by:

\[ p \mapsto [s_1(p) : \cdots : s_n(p)] \in \mathbb{P}^{n-1} \]

Degree $d$ maps correspond to the case $\deg(S^\vee) = d$. 
Background
Maps and Quot Schemes

The moduli space of maps $C \to \mathbb{P}^{n-1}$ sits inside of a Quot scheme.
Background
Maps and Quot Schemes

Let $\mathcal{C}_B \to B$ be a family of curves, $E$ a coherent sheaf on $\mathcal{C}_B$. 
Background

Maps and Quot Schemes

Let $C_B \to B$ be a family of curves, $E$ a coherent sheaf on $C_B$. The Quot scheme $\text{Quot}^{r,d}_{E/C_B/B}$ is defined by:

- A map $f: T \to B$,
- A quotient of $E_T = f^*E$ on $C_T = T \times B C_B$, flat over $T$:

$$E_T \twoheadrightarrow Q \to C_T$$

$$\downarrow$$

$$f \to \downarrow$$

$$T \to \downarrow$$

$$B$$
Background
Maps and Quot Schemes

Let $C_B \to B$ be a family of curves, $E$ a coherent sheaf on $C_B$. The Quot scheme $\text{Quot}_{E/C_B/B}^{r,d}$ is defined by:

$$\{ \left\{ \text{Maps} \right\} \quad \begin{array}{c} T \to \text{Quot}_{E/C_B/B}^{r,d} \\ \end{array} \quad \begin{array}{c} 1-1 \leftrightarrow \end{array} \quad \begin{array}{c} \text{Families of quotients of } E \\ \text{parameterized by } T \\ \text{with rank } r \text{ and relative degree } d \end{array} \}$$

Meaning:
Background
Maps and Quot Schemes

Let $C_B \rightarrow B$ be a family of curves, $E$ a coherent sheaf on $C_B$.
The Quot scheme $\text{Quot}_{E/C_B/B}^{r,d}$ is defined by:

\[
\begin{align*}
\left\{ \begin{array}{c}
\text{Maps} \\
T \rightarrow \text{Quot}_{E/C_B/B}^{r,d}
\end{array} \right\} & \overset{1-1}{\leftrightarrow} \left\{ \begin{array}{c}
\text{Families of quotients of } E \\
\text{parameterized by } T
\end{array} \right\} \\
\end{align*}
\]

Meaning:

(i) A map $f : T \rightarrow B$,
(ii) A quotient of $E_T = \overline{f}^* E$ on $C_T = T \times_B C_B$, flat over $T$:

\[
\begin{align*}
[E_T \rightarrow Q] & \xrightarrow{\sim} C_T \xrightarrow{\overline{f}} C_B \\
T \xrightarrow{f} B
\end{align*}
\]
Background
Maps and Quot Schemes

Example:
Consider $Q_d = \text{Quot}_{\mathcal{O}_{\mathbb{P}^1}/\mathbb{P}^1/\mathbb{C}}^{n-1,d}$. Points correspond to sequences $0 \to S \to \mathcal{O}^{\oplus n}_{\mathbb{P}^1} \to Q \to 0$ with $\deg(S) = d$. $Q$ may not be locally free. Try to define $\mathbb{P}^1 \to \mathbb{P}^{n-1}$: $p \mapsto [s_1(p), \ldots, s_n(p)]$ like before. Now the $s_i$ can all vanish at the same point (rational maps). The degree of $\tau(Q)$ at $p = $ common order of vanishing of $s_i(p)$. 
Example:
Consider \( Q_d = \text{Quot}^{n-1,d}_{\mathcal{O}^{\oplus n}/\mathbb{P}^1/\mathbb{C}} \).

- Points correspond to sequences

\[
0 \to S \to \mathcal{O}^{\oplus n}_{\mathbb{P}^1} \to Q \to 0
\]

with \( \deg(S^\vee) = d \). \( Q \) may not be locally free.
Background
Maps and Quot Schemes

Example:
Consider $Q_d = \text{Quot}^{n-1,d}_{\mathcal{O}_{\mathbb{P}^1}^\oplus n/\mathbb{P}^1/\mathbb{C}}$.

- Points correspond to sequences

\[ 0 \to S \to \mathcal{O}_{\mathbb{P}^1}^\oplus n \to Q \to 0 \]

with $\deg(S^\vee) = d$. $Q$ may not be locally free.

- Try to define $\mathbb{P}^1 \to \mathbb{P}^{n-1} : p \mapsto [s_1(p), \cdots, s_n(p)]$ like before.
Background
Maps and Quot Schemes

Example:
Consider \( Q_d = \text{Quot}^{n-1,d}_{\mathcal{O}_{\mathbb{P}^1}^n/\mathbb{P}^1/\mathbb{C}} \).

- Points correspond to sequences

\[
0 \rightarrow S \rightarrow \mathcal{O}_{\mathbb{P}^1}^n \rightarrow Q \rightarrow 0
\]

with \( \text{deg}(S^\vee) = d \). \( Q \) may not be locally free.

- Try to define \( \mathbb{P}^1 \rightarrow \mathbb{P}^{n-1} : p \mapsto [s_1(p), \cdots, s_n(p)] \) like before.

- Now the \( s_i \) can all vanish at the same point (rational maps).
  The degree of \( \tau(Q) \) at \( p = \text{common order of vanishing of } s_i(p) \).
Background
Maps and Quot Schemes

$Q_d$ is a nonsingular compactification of the space of maps $\mathbb{P}^1 \to \mathbb{P}^{n-1}$.
Background
Maps and Quot Schemes

\( Q_d \) is a nonsingular compactification of the space of maps \( \mathbb{P}^1 \to \mathbb{P}^{n-1} \).

The boundary is singular with high codimension.
Background
Maps and Quot Schemes

$Q_d$ is a nonsingular compactification of the space of maps $\mathbb{P}^1 \to \mathbb{P}^{n-1}$.

The boundary is singular with high codimension.

- The boundary has a filtration

$$Z_{d,0} \hookrightarrow Z_{d,1} \hookrightarrow \cdots \hookrightarrow Z_{d,d-1} \hookrightarrow Q_d$$

where $Z_{d,k} = \{\text{Quotients with degree of torsion } \geq d - k\}$
Background
Maps and Quot Schemes

$Q_d$ is a nonsingular compactification of the space of maps $\mathbb{P}^1 \to \mathbb{P}^{n-1}$.

The boundary is singular with high codimension.

- The boundary has a filtration

$$Z_{d,0} \hookrightarrow Z_{d,1} \hookrightarrow \cdots \hookrightarrow Z_{d,d-1} \hookrightarrow Q_d$$

where $Z_{d,k} = \{\text{Quotients w/ degree of torsion } \geq d - k\}$

- On $\mathbb{P}^1_{Q_d}$, the locus where all $s_i$ vanish has codimension $n$. The image in $Q_d$ is $Z_{d,d-1}$, so the boundary has codimension $n - 1$. 
Background
Maps and Quot Schemes

$Q_d$ is a nonsingular compactification of the space of maps $\mathbb{P}^1 \to \mathbb{P}^{n-1}$.

The boundary is singular with high codimension.

- The boundary has a filtration

$$Z_{d,0} \hookrightarrow Z_{d,1} \hookrightarrow \cdots \hookrightarrow Z_{d,d-1} \hookrightarrow Q_d$$

where $Z_{d,k} = \{ \text{Quotients w/ degree of torsion} \geq d - k \}$

- On $\mathbb{P}^1_{Q_d}$, the locus where all $s_i$ vanish has codimension $n$.
  The image in $Q_d$ is $Z_{d,d-1}$, so the boundary has codimension $n - 1$.

Yijun Shao carried out a blow-up procedure on $Q_d$ yielding a boundary which is a simple normal crossings divisor.
What about the space of degree $d$ maps from a genus 1 curve to $\mathbb{P}^{n-1}$?
What about the space of degree $d$ maps from a genus 1 curve to $\mathbb{P}^{n-1}$?

Marian, Oprea, and Pandharipande define a nonsingular compactification.
Background
(Quasi-)Stable Quotients

What about the space of degree $d$ maps from a genus 1 curve to $\mathbb{P}^{n-1}$?

Marian, Oprea, and Pandharipande define a nonsingular compactification.

- Allow nodal curves,
What about the space of degree $d$ maps from a genus 1 curve to $\mathbb{P}^{n-1}$?

Marian, Oprea, and Pandharipande define a nonsingular compactification.
- Allow nodal curves,
- Allow quotients which are not locally free.
(Quasi-)Stable Quotients

What about the space of degree $d$ maps from a genus 1 curve to $\mathbb{P}^{n-1}$?

Marian, Oprea, and Pandharipande define a nonsingular compactification.

- Allow nodal curves,
- Allow quotients which are not locally free.

This is the **moduli space of stable quotients** $Q_d$. 
Background

(Quasi-)Stable Quotients

What about the space of degree $d$ maps from a genus 1 curve to $\mathbb{P}^{n-1}$?

Marian, Oprea, and Pandharipande define a nonsingular compactification.

- Allow nodal curves,
- Allow quotients which are not locally free.

This is the moduli space of stable quotients $Q_d$.

A nodal curve has singularities that look like $xy = 0$ (analytically).

(Picture: A collection of smooth curves stuck together at some points)
Background

(Quasi-)Stable Quotients

What about the space of degree $d$ maps from a genus 1 curve to $\mathbb{P}^{n-1}$?

Marian, Oprea, and Pandharipande define a nonsingular compactification.

- Allow nodal curves,
- Allow quotients which are not locally free.

This is the **moduli space of stable quotients** $Q_d$.

A **nodal curve** has singularities that look like $xy = 0$ (analytically).

(Picture: A collection of smooth curves stuck together at some points)

A **semi-stable** genus 1 curve is smooth or a cycle of $\mathbb{P}^1$'s.
A \textbf{degree }$d$\textbf{ quasi-stable quotient} is:
(Quasi-)Stable Quotients

A degree $d$ quasi-stable quotient is:
- A semi-stable curve $C$ ($\leftarrow g = 1$),
Background

(Quasi-)Stable Quotients

A degree $d$ quasi-stable quotient is:

- A semi-stable curve $C$ \( \langle g = 1 \rangle \),
- A quotient of $O_C^\oplus n$

$$0 \rightarrow S \rightarrow O_C^\oplus n \rightarrow Q \rightarrow 0$$

satisfying:

Stable means (in this setting):

$$\text{deg} S \vee |C_i| > 0 \text{ for each component } C_i$$

$Q$ and $\tilde{Q}$ denote the moduli space of stable (resp. quasi-stable) quotients.
A degree $d$ quasi-stable quotient is:

- A semi-stable curve $C$ ($\leftarrow g = 1$),
- A quotient of $\mathcal{O}_C^\oplus_n$

\[
0 \to S \to \mathcal{O}_C^\oplus_n \to Q \to 0
\]

satisfying:

(i) $Q$ has rank $n - 1$ and degree $d$,
A degree $d$ quasi-stable quotient is:

- A semi-stable curve $C$ ($\leftarrow g = 1$),
- A quotient of $\mathcal{O}_C^{\oplus n}$

\[
0 \to S \to \mathcal{O}_C^{\oplus n} \to Q \to 0
\]

satisfying:

(i) $Q$ has rank $n - 1$ and degree $d$,
(ii) $Q$ is locally free at the nodes of $C$. 

(Quasi-)Stable Quotients
A degree $d$ quasi-stable quotient is:
- A semi-stable curve $C$ ($\leftarrow g = 1$),
- A quotient of $\mathcal{O}_C^\oplus n$

$$0 \to S \to \mathcal{O}_C^\oplus n \to Q \to 0$$

satisfying:
- (i) $Q$ has rank $n - 1$ and degree $d$,
- (ii) $Q$ is locally free at the nodes of $C$.

**Stable** means (in this setting):
$$\deg S^\vee|_{C_i} > 0$$
for each component $C_i$ of the normalization of $C$.
(Quasi-)Stable Quotients

A degree $d$ quasi-stable quotient is:

- A semi-stable curve $C$ ($\leftarrow g = 1$),
- A quotient of $\mathcal{O}_C^\oplus n$

$$0 \to S \to \mathcal{O}_C^\oplus n \to Q \to 0$$

satisfying:

(i) $Q$ has rank $n - 1$ and degree $d$,
(ii) $Q$ is locally free at the nodes of $C$.

- **Stable** means (in this setting):
  \[ \deg S^\vee \big|_{C_i} > 0 \text{ for each component } C_i \text{ of the normalization of } C \]

$\mathcal{Q}_d$ (resp. $\tilde{\mathcal{Q}}_d$) = moduli space of stable (resp. quasi-stable) quotients.
Background

(Quasi-)Stable Quotients

A few things to note:
A few things to note:

- The inclusion $Q_d \hookrightarrow \tilde{Q}_d$ is an open embedding.
A few things to note:

- The inclusion $Q_d \hookrightarrow \tilde{Q}_d$ is an open embedding.
- There are forgetful maps $\tilde{Q}_d \to M_1$ and $Q_d \to M_1$. (It can be shown that these are smooth)
Background

(Quasi-)Stable Quotients

A few things to note:

- The inclusion $Q_d \hookrightarrow \tilde{Q}_d$ is an open embedding.
- There are forgetful maps $\tilde{Q}_d \to M_1$ and $Q_d \to M_1$.
  (It can be shown that these are smooth)
- For a fixed family of curves $U \to M_1$, there is an open embedding

\[ U \times M_1 \tilde{Q}_d \hookrightarrow \text{Quot}^{n-1,d}_{O_{C_U}^{\oplus n}/C_U/U} \]
Background

(Quasi-)Stable Quotients

A few things to note:

- The inclusion $Q_d \hookrightarrow \tilde{Q}_d$ is an open embedding.
- There are forgetful maps $\tilde{Q}_d \to M_1$ and $Q_d \to M_1$. (It can be shown that these are smooth)
- For a fixed family of curves $U \to M_1$, there is an open embedding

$$U \times M_1 \tilde{Q}_d \hookrightarrow \text{Quot}^{n-1,d}_{O_{C_U}^n/C_U/U}$$

**Theorem** (Marian, Oprea, Pandharipande): $Q_d$ is a nonsingular, irreducible, separated, proper Deligne-Mumford stack of finite type over $\mathbb{C}$, of dimension $nd$. 

Thomas D. Maienschein ()

Desingularizing the Boundary of the Moduli $S$
Background
Blowing up

Like $Q_d (g = 0)$:
Background
Blowing up

Like $Q_d \ (g = 0)$:

- There is a filtration of the boundary

$$\mathcal{Z}_{d,0} \hookrightarrow \mathcal{Z}_{d,1} \hookrightarrow \cdots \hookrightarrow \mathcal{Z}_{d,d-1} \hookrightarrow Q_d$$

where $\mathcal{Z}_{d,k} = \{\text{Quotients w/ degree of torsion} \geq d - k\}$
Like $Q_d\ (g = 0)$:

- There is a filtration of the boundary

  $$Z_{d,0} \hookrightarrow Z_{d,1} \hookrightarrow \cdots \hookrightarrow Z_{d,d-1} \hookrightarrow Q_d$$

  where $Z_{d,k} = \{\text{Quotients w/ degree of torsion} \geq d - k\}$

- The boundary is singular and has high codimension (n-1).
Like $Q_d \ (g = 0)$:

- There is a filtration of the boundary

$$
\mathcal{Z}_{d,0} \hookrightarrow \mathcal{Z}_{d,1} \hookrightarrow \cdots \hookrightarrow \mathcal{Z}_{d,d-1} \hookrightarrow Q_d
$$

where $\mathcal{Z}_{d,k} = \{ \text{Quotients w/ degree of torsion } \geq d - k \}$

- The boundary is singular and has high codimension (n-1).

We will adapt the blow-up process for $Q_d \ (g = 0)$ to $Q_d \ (g = 1)$.

**Goal:** Show the resulting boundary = divisor with simple normal crossings.
Background

Blowing up

Blow up each row along the space indicated by a box.

\[
\begin{array}{ccc}
\mathcal{Z}^{d-1}_{d,0} & \mathcal{Z}^{d-1}_{d,1} & \cdots & \mathcal{Z}^{d-1}_{d,d-1} \rightarrow Q^{d-1}_d \\
\downarrow & \downarrow & & \downarrow \\
\mathcal{Z}^{d-2}_{d,0} & \mathcal{Z}^{d-2}_{d,1} & \cdots & \mathcal{Z}^{d-2}_{d,d-1} \rightarrow Q^{d-2}_d \\
\downarrow & \downarrow & & \downarrow \\
\vdots & \vdots & & \vdots \\
\downarrow & \downarrow & & \downarrow \\
\mathcal{Z}^0_{d,0} & \mathcal{Z}^0_{d,1} & \cdots & \mathcal{Z}^0_{d,d-1} \rightarrow Q^0_d \\
\downarrow & \downarrow & & \downarrow \\
\mathcal{Z}^d_{d,0} & \mathcal{Z}^d_{d,1} & \cdots & \mathcal{Z}^d_{d,d-1} \rightarrow Q^d_d \\
\end{array}
\]

**Theorem:** $\mathcal{Z}^{d-1}_{d,0}, \ldots, \mathcal{Z}^{d-1}_{d,d-1}$ are nonsingular, codimension 1, and intersect transversally in $Q^{d-1}_d$.
Consider the universal sequence on $\mathcal{C}_{Q_d}$:

$$0 \rightarrow S \rightarrow \mathcal{O}_{\mathcal{C}_{Q_d}}^{\oplus n} \rightarrow Q \rightarrow 0$$
The Setup
Defining the Scheme Structure

Consider the universal sequence on $\mathcal{C}_{Q_d}$:

$$0 \to S \to \mathcal{O}_{\mathcal{C}_{Q_d}}^\oplus n \to Q \to 0$$

Dualize, twist, and push down to $Q_d$:

$$\pi_* \mathcal{O}_{\mathcal{C}_{Q_d}}^\oplus n \vee (m) \xrightarrow{\rho^m} \pi_* S^\vee (m)$$

(Stability implies there is a relatively ample bundle for the twisting)
The Setup
Defining the Scheme Structure

Consider the universal sequence on $\mathcal{C}_{Q_d}$:

$$0 \rightarrow S \rightarrow \mathcal{O}_{\mathcal{C}_{Q_d}}^{\oplus n} \rightarrow Q \rightarrow 0$$

Dualize, twist, and push down to $Q_d$:

$$\pi_* \mathcal{O}_{\mathcal{C}_{Q_d}}^{\oplus n} \overset{\rho_m}{\rightarrow} \pi_* S^{\vee} (m)$$

(Stability implies there is a relatively ample bundle for the twisting)

For $m \gg 0$, this is a map of bundles on $Q_d$. 
The Setup
Defining the Scheme Structure

For $q = (C, \mathcal{O}_C \otimes^n \to Q) \in \mathcal{Q}_d$,

$$\text{rank } \rho_m|_q = mD + d - \deg \tau(Q)$$

($D$ is the degree of the ample line bundle on $C_{Q_d}$)
The Setup

Defining the Scheme Structure

For \( q = (C, \mathcal{O}_C^\oplus n \rightarrow Q) \in \mathcal{Q}_d \),

\[
\text{rank } \rho_m|_q = mD + d - \deg \tau(Q)
\]

(\( D \) is the degree of the ample line bundle on \( \mathcal{C}_{\mathcal{Q}_d} \))

Hence:

\[
\deg \tau(Q) \geq d - k \iff \text{rank } \rho_m|_q \leq mD + k
\]
For \( q = (C, \mathcal{O}_C^\oplus n \to Q) \in \mathcal{Q}_d \),

\[
\text{rank } \rho_m|_q = mD + d - \deg \tau(Q)
\]

(\( D \) is the degree of the ample line bundle on \( \mathcal{C}_{Q_d} \))

Hence:

\[
\deg \tau(Q) \geq d - k \iff \text{rank } \rho_m|_q \leq mD + k
\]

Define \( \mathcal{Z}_{d,k} \) to be the vanishing of \( mD + k + 1 \)

\[
\bigwedge^{mD+k+1} \rho_m.
\]
The Setup
Preparing for Induction: Factorizations

To use inductive arguments, we want to relate the degree $d$ procedure to the degree $k$ procedure for $k < d$. 
Idea: A degree $d$ stable quotient in $Z_{d,k}$ can be expressed as a pair:
The Setup
Preparing for Induction: Factorizations

**Idea:** A degree $d$ stable quotient in $\mathcal{Z}_{d,k}$ can be expressed as a pair:

$$0 \to S \xrightarrow{f} \mathcal{O}_C^\oplus n \to F \to 0 \quad (\leftarrow \text{degree } k \text{ quasi-stable quotient})$$
The Setup
Preparing for Induction: Factorizations

**Idea:** A degree $d$ stable quotient in $\mathcal{Z}_{d,k}$ can be expressed as a pair:

\[
\begin{array}{ccc}
0 & \rightarrow & S \\
& \xrightarrow{f} & \mathcal{O}_C^n \\
& & \rightarrow F \\
& & \rightarrow 0 \\
\end{array} \quad (\leftarrow \text{degree } k \text{ quasi-stable quotient})
\]

\[
\begin{array}{ccc}
0 & \rightarrow & K \\
& \xrightarrow{g} & S \\
& & \rightarrow T \\
& & \rightarrow 0 \\
\end{array} \quad (\leftarrow K \text{ is a line bundle of degree } -d)
\]
The Setup
Preparing for Induction: Factorizations

**Idea:** A degree $d$ stable quotient in $\mathcal{Z}_{d,k}$ can be expressed as a pair:

\[
0 \rightarrow S \xrightarrow{f} \mathcal{O}_C^\oplus n \rightarrow F \rightarrow 0 \quad (\leftarrow \text{degree } k \text{ quasi-stable quotient})
\]
\[
0 \rightarrow K \xrightarrow{g} S \rightarrow T \rightarrow 0 \quad (\leftarrow K \text{ is a line bundle of degree } -d)
\]

(Stability condition: $\omega_C \otimes (K^\vee)^\otimes \epsilon$ is ample for all $0 < \epsilon \in \mathbb{Q}$)
Idea: A degree $d$ stable quotient in $\mathcal{Z}_{d,k}$ can be expressed as a pair:

$$
0 \rightarrow S \xrightarrow{f} \mathcal{O}_C^\oplus n \rightarrow F \rightarrow 0 \quad (\leftarrow \text{degree } k \text{ quasi-stable quotient})
$$

$$
0 \rightarrow K \xrightarrow{g} S \rightarrow T \rightarrow 0 \quad (\leftarrow K \text{ is a line bundle of degree } -d)
$$

(Stability condition: $\omega_C \otimes (K^\vee)^{\otimes \epsilon}$ is ample for all $0 < \epsilon \in \mathbb{Q}$)

Form the cokernel of $f \circ g$:

$$
0 \rightarrow K \rightarrow \mathcal{O}_C^\oplus n \rightarrow Q \rightarrow 0 \quad (\leftarrow \text{degree } d \text{ stable quotient})
$$
Idea: A degree $d$ stable quotient in $\mathcal{Z}_{d,k}$ can be expressed as a pair:

$$0 \to S \xrightarrow{f} \mathcal{O}_C^\oplus n \to F \to 0 \quad (\leftarrow \text{degree } k \text{ quasi-stable quotient})$$

$$0 \to K \xrightarrow{g} S \to T \to 0 \quad (\leftarrow K \text{ is a line bundle of degree } -d)$$

(Stability condition: $\omega_C \otimes (K^\vee)^{\otimes \epsilon}$ is ample for all $0 < \epsilon \in \mathbb{Q}$)

Form the cokernel of $f \circ g$:

$$0 \to K \to \mathcal{O}_C^\oplus n \to Q \to 0 \quad (\leftarrow \text{degree } d \text{ stable quotient})$$

It can be shown that $Q$ fits into the short exact sequence:

$$0 \to T \to Q \to F \to 0.$$
The Setup
Preparing for Induction: Factorizations

Define $Q_{d,k}$ to be the moduli space of these pairs ("factorizations").

There are maps:

$$
\tilde{Q}_k \leftarrow \theta \quad \text{forget} \quad Q_{d,k} \xrightarrow{\phi} \text{compose} \quad Q_d
$$

What if we require the degree $k$ quotient to be stable?

Suppose $C$ has $> k$ irreducible components. Then $C$ does not admit a degree $k$ stable quotient.

Conclusion:

To use induction we will have to blow up $\tilde{Q}_d$. 

Thomas D. Maienschein ()
Desingularizing the Boundary of the Moduli S
June 9, 2014
21 / 32
The Setup
Preparing for Induction: Factorizations

Define $Q_{d,k}$ to be the moduli space of these pairs ("factorizations").

There are maps:

$$\tilde{Q}_k \leftarrow \theta \begin{array}{c} \text{forget} \\ \text{compose} \end{array} Q_{d,k} \rightarrow \phi Q_d$$

What if we require the degree $k$ quotient to be stable?
Define $Q_{d,k}$ to be the moduli space of these pairs ("factorizations").

There are maps:

$$
\tilde{Q}_k \leftarrow Q_{d,k} \xrightarrow{\theta} Q_d. 
$$

What if we require the degree $k$ quotient to be stable?

- Suppose $C$ has $> k$ irreducible components.
The Setup
Preparing for Induction: Factorizations

Define $Q_{d,k}$ to be the moduli space of these pairs ("factorizations").

There are maps:

$$
\tilde{Q}_k \leftarrow \theta \text{ forget} \quad Q_{d,k} \xrightarrow{\phi \text{ compose}} Q_d
$$

What if we require the degree $k$ quotient to be stable?

- Suppose $C$ has $> k$ irreducible components.
- Then $C$ does not admit a degree $k$ stable quotient.
The Setup
Preparing for Induction: Factorizations

Define \( Q_{d,k} \) to be the moduli space of these pairs (“factorizations”).

There are maps:

\[
\begin{align*}
\tilde{Q}_k & \overset{\theta}{\longrightarrow} Q_{d,k} \\
& \overset{\text{forget}}{\leftarrow} Q_{d,k} \overset{\phi}{\longrightarrow} Q_d
\end{align*}
\]

What if we require the degree \( k \) quotient to be stable?

- Suppose \( C \) has \( > k \) irreducible components.
- Then \( C \) does not admit a degree \( k \) stable quotient.
- \( \implies \) No degree \( d \) stable quotient on \( C \) can be “factored”. 

Conclusion: To use induction we will have to blow up \( \tilde{Q}_d \).
Define $Q_{d,k}$ to be the moduli space of these pairs ("factorizations").

There are maps:

$$\tilde{Q}_k \leftarrow \theta \ Q_{d,k} \rightarrow \phi \ Q_d$$

What if we require the degree $k$ quotient to be stable?

- Suppose $C$ has $> k$ irreducible components.
- Then $C$ does not admit a degree $k$ stable quotient.

$\implies$ No degree $d$ stable quotient on $C$ can be "factored".

**Conclusion:** To use induction we will have to blow up $\tilde{Q}_d$. 
The Setup

Working smooth-locally

**Problem:** What do we blow up on $\tilde{Q}_d$?
The Setup

Working smooth-locally

**Problem:** What do we blow up on $\tilde{Q}_d$?

- Stability $\implies$ relatively ample line bundle on $C_{Q_d}$.
The Setup

Working smooth-locally

**Problem:** What do we blow up on $\tilde{Q}_d$?

- Stability $\iff$ relatively ample line bundle on $C_{Q_d}$.
- Used when defining $Z_{d,k}$.
The Setup

Working smooth-locally

**Problem:** What do we blow up on $\tilde{Q}_d$?

- Stability $\iff$ relatively ample line bundle on $C_{Q_d}$.
- Used when defining $Z_{d,k}$.
- No relatively ample line bundle on $C_{\tilde{Q}_d}$.
The Setup

Working smooth-locally

**Problem:** What do we blow up on $\tilde{Q}_d$?

- Stability $\implies$ relatively ample line bundle on $C_{Q_d}$.
- Used when defining $Z_{d,k}$.
- No relatively ample line bundle on $C_{\tilde{Q}_d}$.

**Solution:** Cover $M_1$ by smooth $U \to M_1$ with $C_U \to U$ projective:

$$(\exists \text{ rel. ample line bundle } \implies) \quad C_U \to C_{M_1}$$

$$(\text{Open in Quot}_{n-1,d}^{\mathcal{O}_{C_U}/C_U/U}) \quad C_{U_d} \to U$$

$$(\exists \text{ rel. ample line bundle } \implies) \quad U_d \to \tilde{Q}_d \to U \to M_1$$

Thomas D. Maienschein

Desingularizing the Boundary of the Moduli Space of Genus One Stable Quotients

June 9, 2014
The Setup
Working smooth-locally

Use the relatively ample line bundle on $C_{U_d}$ to define

$$V_{d,0} \hookrightarrow V_{d,1} \hookrightarrow \cdots \hookrightarrow V_{d,d-1} \hookrightarrow U_d.$$
The Setup
Working smooth-locally

Use the relatively ample line bundle on $C_{U_d}$ to define

$$V_{d,0} \hookrightarrow V_{d,1} \hookrightarrow \cdots \hookrightarrow V_{d,d-1} \hookrightarrow U_d.$$ 

Consider the universal sequence on $C_{U_d}$:

$$0 \to S_U \to \mathcal{O}_{C_{U_d}}^\oplus n \to \mathcal{Q}_U \to 0$$
The Setup

Working smooth-locally

Use the relatively ample line bundle on $\mathcal{C}_{U_d}$ to define

\[ V_{d,0} \hookrightarrow V_{d,1} \hookrightarrow \cdots \hookrightarrow V_{d,d-1} \hookrightarrow U_d. \]

Consider the universal sequence on $\mathcal{C}_{U_d}$:

\[ 0 \rightarrow S_U \rightarrow \mathcal{O}_{\mathcal{C}_{U_d}}^\oplus n \rightarrow \mathcal{Q}_U \rightarrow 0 \]

Dualize, twist, and push down to $\mathcal{Q}_d$:

\[ \pi_* \mathcal{O}_{\mathcal{C}_{U_d}}^\oplus n \vee (m) \xrightarrow{\rho^m} \pi_* S_U \vee (m) \]
The Setup

Working smooth-locally

Use the relatively ample line bundle on $C_{U_d}$ to define

$$V_{d,0} \hookrightarrow V_{d,1} \hookrightarrow \cdots \hookrightarrow V_{d,d-1} \hookrightarrow U_d.$$ 

Consider the universal sequence on $C_{U_d}$:

$$0 \to S_U \to \mathcal{O}_{C_{U_d}}^{\oplus n} \to Q_U \to 0$$

Dualize, twist, and push down to $Q_d$:

$$
\pi_* \mathcal{O}_{C_{U_d}}^{\oplus n} \vee (m) \xrightarrow{\rho^m} \pi_* S_U \vee (m)
$$

Define $V_{d,k}$ to be the vanishing of $mD + k + 1$ of $\bigwedge \rho_m$. 

Define $V_{d,k}$ to be the vanishing of $mD + k + 1$ of $\bigwedge \rho_m$. 

Define $V_{d,k}$ to be the vanishing of $mD + k + 1$ of $\bigwedge \rho_m$. 

Define $V_{d,k}$ to be the vanishing of $mD + k + 1$ of $\bigwedge \rho_m$.
The Setup
Working smooth-locally

The $V_{d,k}$ should be closely related to the $\mathcal{Z}_{d,k}$.
The Setup
Working smooth-locally

The $V_{d,k}$ should be closely related to the $\mathcal{Z}_{d,k}$.

\[
\begin{align*}
U_d^\dagger & \xrightarrow{j} Q_d \\
i & \searrow \nwarrow \downarrow \downarrow \Downarrow \nwarrow \nwarrow \\
U & \xrightarrow{\text{smooth}} \tilde{Q}_d \\
U_d & \xrightarrow{\text{smooth}} \tilde{Q}_d \\
\end{align*}
\]

It can be shown that $i$ is an open embedding and $j$ is smooth, $i^{-1}(V_{d,k}) = j^{-1}(Z_{d,k})$ (Different bundles used for twisting)
The Setup

Working smooth-locally

The \( V_{d,k} \) should be closely related to the \( \bar{Z}_{d,k} \).

\[
\begin{array}{ccc}
U_d^+ & \xrightarrow{j} & Q_d \\
\downarrow{i} & & \downarrow{\tilde{Q}_d}
\end{array}
\]

\[
\begin{array}{ccc}
U & \xrightarrow{\text{smooth}} & \bar{Q}_d \\
\downarrow{i} & & \downarrow{\tilde{Q}_d}
\end{array}
\]

\[\xrightarrow{M_1}\]

It can be shown that
The Setup

Working smooth-locally

The $V_{d,k}$ should be closely related to the $Z_{d,k}$.

$$\xymatrix{ U^\dagger_d \ar[r]^j \ar[d]^i & Q_d \ar[d]^U \ar[r] & M_1 \ar[d] \ar[r] & \\
U_d \ar[r]_{\text{smooth}} & Q_d \ar[r] & }$$

It can be shown that

- $i$ is an open embedding and $j$ is smooth,
The Setup

Working smooth-locally

The $V_{d,k}$ should be closely related to the $\mathcal{Z}_{d,k}$.

\[
\begin{array}{ccc}
U_d^\dagger & \xrightarrow{j} & Q_d \\
i & & \downarrow \\
U & \xrightarrow{\text{smooth}} & \mathcal{M}_1 \\
i & & \\
U_d & \xrightarrow{} & \mathcal{Q}_d \\
\end{array}
\]

It can be shown that

- $i$ is an open embedding and $j$ is smooth,
- $i^{-1}(V_{d,k}) = j^{-1}(\mathcal{Z}_{d,k})$ (← Different bundles used for twisting)
The Setup

Working smooth-locally

The $V_{d,k}$ should be closely related to the $Z_{d,k}$.

\[
\begin{array}{c}
U_d^\dagger \\
\downarrow i \\
U \\
\downarrow \text{smooth} \\
U_d
\end{array} \quad \xrightarrow{j} \quad \begin{array}{c}
Q_d \\
\downarrow \text{smooth} \\
\tilde{Q}_d
\end{array}
\]

Hence:
The Setup
Working smooth-locally

The $V_{d,k}$ should be closely related to the $Z_{d,k}$.

Hence:

- $j$ induces smooth maps $\tilde{j} : (U_d^r)^\dagger \rightarrow Q_d^r$, 

\[ \begin{array}{ccc} 
U_d^\dagger & \xrightarrow{j} & Q_d \\
| & | & | \\
| & i & | \\
U & \xrightarrow{\text{smooth}} & \tilde{Q}_d \\
| & | & | \\
| & | & | \\
U_d & \xrightarrow{j} & Q_d \\
\end{array} \]
The Setup
Working smooth-locally

The $V_{d,k}$ should be closely related to the $Z_{d,k}$.

\[
\begin{array}{ccc}
U_d^\dagger & \xrightarrow{j} & Q_d \\
\downarrow i & & \downarrow Q_d \\
U & \xrightarrow{\text{smooth}} & M_1 \\
\downarrow U_d & & \downarrow \tilde{Q}_d \\
\end{array}
\]

Hence:

- $j$ induces smooth maps $\tilde{j} : (U_d^r)^\dagger \rightarrow Q_d^r$,
- $\tilde{j}^{-1}(Z_{d,k}^r) = (V_{d,k}^r)^\dagger$. 
The Setup
Working smooth-locally

The $V_{d,k}$ should be closely related to the $Z_{d,k}$.

Hence:

- $j$ induces smooth maps $\tilde{j} : (U_d^r)\dagger \rightarrow Q_d^r$,
- $\tilde{j}^{-1}(Z_{d,k}^r) = (V_{d,k}^r)\dagger$.
- If $P$ is smooth-local, $\{V_{d,k}^{d-1}\}$ satisfy $P \implies \{Z_{d,k}^{d-1}\}$ satisfy $P$. 
The Setup
Working smooth-locally

The blow-up process on $U_d$:

\[
\begin{align*}
V_{d,0}^{d-1} & \rightarrow V_{d,1}^{d-1} & \cdots & \rightarrow V_{d,d-1}^{d-1} & \hookrightarrow U_{d-1}^d \\
\downarrow & \downarrow & \cdots & \downarrow & \downarrow \\
V_{d,0}^{d-2} & \rightarrow V_{d,1}^{d-2} & \cdots & \rightarrow V_{d,d-1}^{d-2} & \hookrightarrow U_{d-2}^d \\
\downarrow & \downarrow & \cdots & \downarrow & \downarrow \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
V_{d,0}^0 & \rightarrow V_{d,1}^0 & \cdots & \rightarrow V_{d,d-1}^0 & \hookrightarrow U_0^d \\
\downarrow & \downarrow & \cdots & \downarrow & \downarrow \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
V_{d,0} & \rightarrow V_{d,1} & \cdots & \rightarrow V_{d,d-1} & \hookrightarrow U_0^d
\end{align*}
\]
The Setup
Working smooth-locally

Back to factorizations:
The Setup

Working smooth-locally

Back to factorizations:

- Drop the stability conditions in $Q_{d,k}$ to obtain $\tilde{Q}_{d,k}$. 
The Setup

Working smooth-locally

Back to factorizations:

- Drop the stability conditions in $Q_{d,k}$ to obtain $\tilde{Q}_{d,k}$.
- Define $U_{d,k} = U \times M_1 \tilde{Q}_{d,k}$ (open in $\text{Quot}^{0,d-k}_{SU/CU_k/U_k}$)
The Setup
Working smooth-locally

Back to factorizations:

- Drop the stability conditions in $Q_{d,k}$ to obtain $\widetilde{Q}_{d,k}$.
- Define $U_{d,k} = U \times \mathcal{M}_1 \widetilde{Q}_{d,k}$ (← Open in $\text{Quot}_{\mathcal{S}_U/\mathcal{C}_U/k/U_k}^{0,d-k}$)

There are maps:

$$
\begin{array}{ccc}
U_k & \xleftarrow{\theta_U} & U_{d,k} \\
\text{forget} & \downarrow & \text{compose} \\
& \phi_U & \rightarrow U_d
\end{array}
$$
The Setup

Working smooth-locally

Back to factorizations:

- Drop the stability conditions in $Q_{d,k}$ to obtain $\tilde{Q}_{d,k}$.
- Define $U_{d,k} = U \times M_1 \tilde{Q}_{d,k}$ (← Open in Quot_{SU/CUk/Uk}^{0,d-k})

There are maps:

$$U_k \xleftarrow{\theta_U \text{ forget}} U_{d,k} \xrightarrow{\phi_U \text{ compose}} U_d$$

We will show that:
The Setup

Working smooth-locally

Back to factorizations:

- Drop the stability conditions in $Q_{d,k}$ to obtain $\tilde{Q}_{d,k}$.
- Define $U_{d,k} = U \times M_1 \tilde{Q}_{d,k}$ (Open in $\text{Quot}_{S_U/C_{U_k}/U_k}^{0,d-k}$)

There are maps:

$$
U_k \xleftarrow{\theta_U} U_{d,k} \xrightarrow{\phi_U} U_d
$$

We will show that:

$$
\check{V}_{d,k} \simeq \check{U}_{d,k} \quad (\leftarrow U_{d,k} \times U_k \check{U}_k)
$$
The Setup

Working smooth-locally

Back to factorizations:

- Drop the stability conditions in $Q_{d,k}$ to obtain $\tilde{Q}_{d,k}$.
- Define $U_{d,k} = U \times M_1 \tilde{Q}_{d,k}$ (← Open in $\text{Quot}_{S_U/C_{U_k}/U_k}^{0,d-k}$)

There are maps:

\[
\begin{array}{ccc}
U_k & \xrightarrow{\theta_U} & U_{d,k} & \xrightarrow{\phi_U} & U_d \\
\text{forget} & & \text{compose} & & \\
\end{array}
\]

We will show that:

\[
\begin{align*}
\check{V}_{d,k} & \simeq \check{U}_{d,k} \quad (\leftarrow U_{d,k} \times U_k \check{U}_k) \\
V_{d,k}^{k-1} & \simeq U_{d,k} \times U_k U_k^{k-1}
\end{align*}
\]
More Details

Space of Collineations

We can embed $U_d$ into a space of collineations. Given bundles $E, F$ on $X$, define $S(E, F) = \mathbb{P}(\mathcal{H}om(E, F))$. 

Vainsencher carries out a blow-up procedure on $S(E, F)$. It exactly corresponds to our blow-up procedure when restricted to $U_d$. Results of Vainsencher imply $U_{k-1}d = \text{Bl}_{b_{k-1}}(V, kU_{k-1}d)$. 
More Details

Space of Collineations

We can embed $U_d$ into a space of collineations. Given bundles $E, F$ on $X$, define $S(E, F) = \mathbb{P}(\mathcal{H}om(E, F))$.

Recall $\text{rank } \rho_m|_q = mD + d - \deg \tau(Q)$. 
More Details

Space of Collineations

We can embed $U_d$ into a space of collineations. Given bundles $E, F$ on $X$, define $\mathbb{S}(E, F) = \mathbb{P}(\mathcal{H}om(E, F))$.

Recall $\text{rank } \rho_m|_q = mD + d - \deg \tau(Q)$. So there is a graph

$$[\rho_m] : U_d \hookrightarrow \mathbb{S} \left( \pi_* \mathcal{O}_{\mathcal{C}_{U_d}}^{\oplus n} \vee (m), \pi_* \mathcal{S}_U \vee (m) \right).$$
More Details

Space of Collineations

We can embed $U_d$ into a space of collineations. Given bundles $E, F$ on $X$, define $\mathbb{S}(E, F) = \mathbb{P}(\mathcal{H}om(E, F))$.

Recall $\text{rank } \rho_m|_q = mD + d - \text{deg } \tau(Q)$.

So there is a graph

$$[\rho_m] : U_d \hookrightarrow \mathbb{S} \left( \pi_* \mathcal{O}_{\mathcal{C}_{U_d}}(m), \pi_* S_U^\vee(m) \right).$$

Vainsencher carries out a blow-up procedure on $\mathbb{S}(E, F)$. It exactly corresponds to our blow-up procedure when restricted to $U_d$. 
More Details

Space of Collineations

We can embed $U_d$ into a space of collineations. Given bundles $E, F$ on $X$, define $\mathbb{S}(E, F) = \mathbb{P}(\mathcal{H}om(E, F))$.

Recall $\text{rank } \rho_m|_q = mD + d - \deg \tau(Q)$. So there is a graph

$$[\rho_m] : U_d \hookrightarrow \mathbb{S} \left( \pi_* \mathcal{O}_{C_{U_d}}^\oplus \overset{\vee}{\wedge} (m), \pi_* \mathcal{S}_U \overset{\vee}{\wedge} (m) \right).$$

Vainsencher carries out a blow-up procedure on $\mathbb{S}(E, F)$. It exactly corresponds to our blow-up procedure when restricted to $U_d$.

Results of Vainsencher $\implies U_d^k \left( = \text{Bl}_{V_{d,k}} U_{d}^{k-1} \right) = \text{Bl}_{b^{-1}(V_{d,k})} U_{d}^{k-1}$.
More Details

Space of Collineations

On $U_d \setminus V_{d,k}$, $\text{rank } \rho_m > mD + k$. 
More Details
Space of Collineations

On $U_d \setminus V_{d,k}$, $\text{rank} \rho_m > mD + k$.

So $\bigwedge_{mD+i} \rho_m$ does not vanish for $i = 0, \ldots, k$. 
More Details
Space of Collineations

On $U_d \setminus V_{d,k}$, \( \text{rank } \rho_m > mD + k \).

So $\bigwedge mD+i \rho_m$ does not vanish for $i = 0, \ldots, k$.

The product of the graphs of $\bigwedge mD+i \rho_m$ give an embedding:

\[
U_d \setminus V_{d,k} \hookrightarrow \prod_{i=0}^{k} U_d \mathbb{S} \left( \bigwedge^{mD+i+1} \pi_* \mathcal{O}_{\mathcal{C}_{U_d}}^\oplus \mathcal{V}(m), \bigwedge^{mD+i+1} \pi_* \mathcal{S}_{U} \mathcal{V}(m) \right)
\]
More Details

Space of Collineations

On $U_{d} \setminus V_{d,k}$, $\text{rank } \rho_{m} > mD + k$.

So $\bigwedge^{mD+i} \rho_{m}$ does not vanish for $i = 0, \ldots, k$.

The product of the graphs of $\bigwedge^{mD+i} \rho_{m}$ give an embedding:

$$ U_{d} \setminus V_{d,k} \hookrightarrow \prod_{i=0}^{k} U_{d} S^{mD+i+1} \left( \bigwedge \pi_{*} \mathcal{O}_{C_{U_d}}^{\oplus n} \vee (m), \bigwedge \pi_{*} \mathcal{S}_{U} \vee (m) \right) $$

Using the result of Vainsencher, the closure of the image is $U_{d}^{k}$. 
To show $V_{d,k}^{k-1} \cong U_{d,k} \times_{U_k} U_k^{k-1}$, construct a commutative diagram:
To show $V_{d,k}^{k-1} \simeq U_{d,k} \times U_k U_k^{k-1}$, construct a commutative diagram:

\[
\begin{align*}
\hat{U}_{d,k} & \xrightarrow{\alpha} \prod_{i=0}^{k-1} U_{d,k} \mathbb{S} \left( \bigwedge_{i=0}^{mD+i+1} \pi_* \mathcal{O}_{\mathcal{C}_{U_{d,k}}}^\bigvee(m), \bigwedge_{i=0}^{mD+i+1} \pi_* \mathcal{S}_U^\bigvee(m) \right) \\
\varphi_U & \downarrow \\
\hat{V}_{d,k} & \xrightarrow{\gamma} \prod_{i=0}^{k-1} U_d \mathbb{S} \left( \bigwedge_{i=0}^{mD+i+1} \pi_* \mathcal{O}_{\mathcal{C}_{U_d}}^\bigvee(m), \bigwedge_{i=0}^{mD+i+1} \pi_* \mathcal{S}_U^\bigvee(m) \right)
\end{align*}
\]
More Details

Beta Diagram

To show $V_{d,k}^{k-1} \simeq U_{d,k} \times U_k U_k^{k-1}$, construct a commutative diagram:

$$
\begin{array}{ccc}
\tilde{U}_{d,k} & \xrightarrow{\alpha} & \prod_{i=0}^{k-1} U_{d,k} \subseteq \left( \bigwedge_{i=0}^{mD+i+1} \pi_\ast \mathcal{O}_{C_{U_d,k}}^n \vee (m), \bigwedge_{i=0}^{mD+i+1} \pi_\ast \mathcal{S}_U \vee (m) \right) \\
\varphi_U & & \\
\tilde{V}_{d,k} & \xrightarrow{\gamma} & \prod_{i=0}^{k-1} U_d \subseteq \left( \bigwedge_{i=0}^{mD+i+1} \pi_\ast \mathcal{O}_{C_{U_d,k}}^n \vee (m), \bigwedge_{i=0}^{mD+i+1} \pi_\ast \mathcal{S}_U \vee (m) \right) \\
\end{array}
$$

- The closure of the image of $\alpha$ is $U_{d,k} \times U_k U_k^{k-1}$,
More Details

Beta Diagram

To show $V^{k-1}_{d,k} \simeq U_{d,k} \times U_k U^{k-1}_k$, construct a commutative diagram:

$$
\begin{align*}
\hat{U}_{d,k} & \xleftarrow{\alpha} \prod_{i=0}^{k-1} U_{d,k} S \left( \bigwedge (mD+i+1) \pi_* O_{C_{U_{d,k}}} \bigvee (m), \bigwedge \pi_* S U^{\bigvee} (m) \right) \\
\varphi_U & \downarrow \quad \beta \\
\hat{V}_{d,k} & \xleftarrow{\gamma} \prod_{i=0}^{k-1} U_d S \left( \bigwedge (mD+i+1) \pi_* O_{C_{U_d}} \bigvee (m), \bigwedge \pi_* S U^{\bigvee} (m) \right)
\end{align*}
$$

- The closure of the image of $\alpha$ is $U_{d,k} \times U_k U^{k-1}_k$,
- The closure of the image of $\gamma$ is $V^{k-1}_{d,k}$,

Thomas D. Maienschein

Desingularizing the Boundary of the Moduli $S$

June 9, 2014 30 / 32
To show $V_{d,k}^{k-1} \simeq U_{d,k} \times U_k U_k^{k-1}$, construct a commutative diagram:

\[
\begin{align*}
\hat{U}_{d,k} & \xrightarrow{\alpha} \prod_{i=0}^{k-1} U_{d,k} \mathbb{S} \left( \bigwedge (mD+i+1) \pi_* O_{C_{U_{d,k}}}^n \lor (m), \bigwedge (mD+i+1) \pi_* S U \lor (m) \right) \\
\hat{V}_{d,k} & \xrightarrow{\gamma} \prod_{i=0}^{k-1} U_d \mathbb{S} \left( \bigwedge (mD+i+1) \pi_* O_{C_{U_d}}^n \lor (m), \bigwedge (mD+i+1) \pi_* S U \lor (m) \right)
\end{align*}
\]

- The closure of the image of $\alpha$ is $U_{d,k} \times U_k U_k^{k-1}$,
- The closure of the image of $\gamma$ is $V_{d,k}^{k-1}$,
- $\beta$ is a closed embedding.
Some things to do:

1. Study the locus of singular curves in $Q^{d-1}$.
2. Is it a nonsingular divisor that intersects the $Z_k^{d-1}$, $k$ transversally?
3. Provide a modular interpretation: Is $Q^{d-1}$ a moduli space of stable quotients + extra data?
4. Is this useful for $g > 1$? The moduli of stable quotients is singular.
Some things to do:

- Study the locus of singular curves in $Q_d^{d-1}$. Is it a nonsingular divisor that intersects the $Z_{d,k}^{k-1}$ transversally?
Future Work

Some things to do:

- Study the locus of singular curves in $Q_d^{d-1}$.
  Is it a nonsingular divisor that intersects the $Z_{d,k}^{k-1}$ transversally?
- Provide a modular interpretation:
  Is $Q_d^{d-1}$ a moduli space of stable quotients + extra data?
Future Work

Some things to do:

- Study the locus of singular curves in $Q_d^{d-1}$.
  Is it a nonsingular divisor that intersects the $Z_{d,k}^{k-1}$ transversally?
- Provide a modular interpretation:
  Is $Q_d^{d-1}$ a moduli space of stable quotients + extra data?
- Is this useful for $g > 1$? The moduli of stable quotients is singular.
References

- Y. Shao, *A compactification of the space of parameterized rational curves in Grassmannians*.
- I. Vainsencher, *Complete collineations and blowing up determinantal ideals*.