Visualizing piecewise-flat manifolds

Thomas Maienschein

November 22, 2010
Background
Definitions

We will call a convex \( n \)-polytope in \( \mathbb{R}^n \) a **facet**. The polytopes forming the boundary of a facet will be called **ridges**. If \( R \) is a ridge of a facet \( F \), we will write \( R < F \).
We will call a convex $n$-polytope in $\mathbb{R}^n$ a **facet**. The polytopes forming the boundary of a facet will be called **ridges**. If $R$ is a ridge of a facet $F$, we will write $R < F$.

Our data will be

- A finite collection of facets,
- A rule for gluing facets along ridges.

The gluing is specified in the following way:
Background

Definitions

To each pair \((F, R)\) with \(R < F\), there is

- A pair \((F', R') = *(F, R)\) with \(R' < F'\)
- An isometry \(T = T(F, R)\) of \(\mathbb{R}^n\)

The space \(X = \text{disjoint union of facets} / \sim\) should be a manifold.

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The following should be an equivalence relation:

\((F, p) \sim (F', p')\) if

- \(p \in R < F\) and \(p' \in R' < F'\), and \(*(F, R) = (F', R')\)
- \(T(F, R)p = p'\)
Background

Definitions

To each pair $(F, R)$ with $R < F$, there is

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The space $X = \text{(disjoint union of facets)}/ \sim$ should be a manifold.
Background
Definitions

If \( \ast(F, R) = (F', R') \), denote the facet \( F' \) by \( \mathcal{F}(F, R) \).
We will identify ridges \( R < F \) and \( R' < F' \) if \( \ast(F, R) = (F', R') \).
If $\ast(F, R) = (F', R')$, denote the facet $F'$ by $\mathcal{F}(F, R)$.
We will identify ridges $R < F$ and $R' < F'$ if $\ast(F, R) = (F', R')$.

Points are pairs $(F, p)$ with $p \in F$, but we may just refer to $p$ if $F$ is understood from context. We will say $p$ is “in the coordinates of $F$”.
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Points are pairs $(F, p)$ with $p \in F$,
but we may just refer to $p$ if $F$ is understood from context.
We will say $p$ is “in the coordinates of $F$”.

We will call points on the boundary of a ridge warped points.
$X \setminus \{\text{warped points}\}$ is a flat Riemannian manifold.
This gives a notion of geodesics and an exponential map.
Geodesics

Example on an embedded surface

Figure: A view from the embedding.
Geodesics
Example: Step 1

Given \( \nu \in T_p F \), we can compute \( \exp_p(\nu) \) in the following way.
Geodesics

Example: Step 1

Given $\nu \in T_pF$, we can compute $\exp_p(\nu)$ in the following way. Let $q = p + \nu$.

**Figure:** Since $q \notin F$, apply $T(F, R)$ to $p$ and $q$. 
Figure: Since $q_1 \notin F_1$, apply $T(F_1, R_1)$ to $p_1$ and $q_1$. 
Geodesics

Example: Step 3

Figure: Since $q_2 \in F_2$, stop; $\exp(v) = (F_2, q_2)$. 
Geodesics

Algorithm

Let $v \in T_p F$.
To find $\exp(v)$, let $q = p + v$, and call the following algorithm:
Let $\nu \in T_pF$.
To find $\exp(\nu)$, let $q = p + \nu$, and call the following algorithm:

**Algorithm**

```
exp(F, p, q) {
  if (q ∈ F) {
    return (F, q);
  }
  else {
    R = Ridge through which the line segment pq exits F;
    F' = \mathcal{F}(F, R);
    T = \mathcal{T}(F, R);
    return exp(F', Tp, Tq);
  }
}
```
Geodesics
Some useful remarks

We saw that for $\nu \in T_p F$, $\exp_p(\nu) = T(p + \nu)$ for some isometry $T$. If we trace the path $\exp_p(t\nu)$ as $t$ goes from 0 to 1, it would pass through some sequence of facets and ridges. This sequence determines the transformation $T$. 
Geodesics
Some useful remarks

We saw that for \( v \in T_p F, \exp_p(v) = T(p + v) \) for some isometry \( T \). If we trace the path \( \exp_p(tv) \) as \( t \) goes from 0 to 1, it would pass through some sequence of facets and ridges. This sequence determines the transformation \( T \).

We can “parallel transport” vectors along \( \exp_p(tv) \) from \( p \) to \( \exp_p(v) \): If \( \exp_p(v) = T(p + v) \), then for \( w \in T_p F \), define

\[
P_v(w) = T(p + w) - T(p) = T(w)
\]
Geodesics
Some useful remarks

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$$P_v(w) = T(p + w) - T(p) = T(w)$$

Now we can move at a consistent velocity in our space. If $w \in T_p F$ is the velocity, and $\Delta t$ seconds have passed,

- Set the new position to $\exp_p(w \cdot \Delta t)$
- Set the new velocity to $P_{w \cdot \Delta t}(w)$. 
Geodesics

Passing through warped points?

How does the algorithm work in the following case?
Geodesics
Passing through warped points?

How does the algorithm work in the following case?

We usually say that $v$ is not in the domain of $\exp$. 
Geodesics

Passing through warped points?

An alternative (in 2-dimensions) is the “straightest geodesic”.

The path is continued in such a way that

\[
\text{(sum of red angles)} = \text{(sum of black angles)}
\]

This can be computed by “walking” around the vertex.
Now we can figure out what we would “see” from a position $p$. Let’s see what kind of phenomena we can expect to occur.
Figure: \((\kappa > 0)\) I see two copies of \(q\) from \(p\).
Geodesics
First step to visualization

Figure: $(\kappa < 0)$ I cannot see $q$ from $p$. 
Fix a facet $F_0$ and a point $p \in F_0$.
We will call $T_pF_0$ the **visual field** for an observer at $p$.
If the observer looks in a direction and distance $\nu \in T_pF_0$, what (s)he sees is whatever is at the point $\exp(\nu)$. 
Fix a facet $F_0$ and a point $p \in F_0$. We will call $T_p F_0$ the **visual field** for an observer at $p$. If the observer looks in a direction and distance $\nu \in T_p F_0$, what (s)he sees is whatever is at the point $\exp(\nu)$.

So, we want to identify $T_p F_0 \simeq \mathbb{R}^n$ and “populate” the visual field by putting at $\nu$ whatever is in the space at $\exp(\nu)$. We can’t do this point-by-point, but we will make some observations allowing us to do this efficiently.
For $v$ in the domain of $\exp$, consider the path $\exp(tv)$ for $t \in [0, 1]$. The path passes through some sequence of facets and ridges.

**Figure:** Here, the sequence is $(F_0, R_0, F_1, R_1, F_2)$. 
Visualization

Observation 1

For $v$ in the domain of $\exp$, consider the path $\exp(tv)$ for $t \in [0, 1]$. The path passes through some sequence of facets and ridges.

![Diagram](image-url)

**Figure:** Here, the sequence is $(F_0, R_0, F_1, R_1, F_2)$.

All $v$ in the domain of $\exp$ have such a sequence $S(v)$. The domain of $\exp$ can be partitioned into sets $\mathcal{D}(S) = \{v | S(v) = S\}$. 
On each of the sets $\mathcal{D}(S)$, $\exp$ has a simple form:
For $S = (F_0, R_0, \ldots, F_k, R_k, F)$ and $v \in \mathcal{D}(S)$,

$$\exp(v) = (\mathcal{T}(F_k, R_k) \circ \cdots \circ \mathcal{T}(F_0, R_0))(p + v).$$
Visualization

Observation 2

On each of the sets $\mathcal{D}(S)$, $\exp$ has a simple form:
For $S = (F_0, R_0, \ldots, F_k, R_k, F)$ and $v \in \mathcal{D}(S)$,

$$\exp(v) = (\mathcal{T}(F_k, R_k) \circ \cdots \circ \mathcal{T}(F_0, R_0))(p + v).$$

Let $\mathcal{E}(S) = \exp(\mathcal{D}(S)) \subset F$.
We can draw the whole chunk $\mathcal{D}(S)$ of the visual field at once, by applying the inverse of the above isometry to $\mathcal{E}(S)$ (and any objects within).
The collection of sequences forms a tree, $\Gamma$:

- The root of the tree is $(F_0)$.
- The children of $(\ldots, F)$ are the sequences $(\ldots, F, R_i, F_i)$, where $R_i < F$ and $F_i = F(F, R_i)$. 

If we can find $E(S')$ for children $S'$ of $S$, where $E(S)$ is known, then by traversing $\Gamma$ we can draw the entire visual field.
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- The root of the tree is $(F_0)$.
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If we can find $E(S')$ for children $S'$ of $S$, where $E(S)$ is known, then by traversing $\Gamma$ we can draw the entire visual field.
The fourth observation is that the sets $\mathcal{E}(S)$ have a particularly nice form. To state the observation, we make the following definition:

**Definition (Frustum)**

A frustum is a subset $V \subseteq \mathbb{R}^n$ such that there exists a source point $p \in \mathbb{R}^n$ and a convex $(n-1)$-polytope $Q \not\ni p$, such that $V = \{ p + k(q - p) \mid q \in Q, k \geq 0 \}$.

It will be convenient to consider $\emptyset$ and $\mathbb{R}^n$ to be frustums (the empty frustum and full frustum, respectively).
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Frustums

Definition

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It will be convenient to consider $\emptyset$ and $\mathbb{R}^n$ to be a frustums (the **empty frustum** and **full frustum**, respectively).
Figure: Examples of frustums.
A frustum can be created from a point and a ridge.
A frustum can be transformed along a ridge.
Frustums can be intersected to get a new frustum (which may be empty).
A frustum $V$ with source point $p$ intersects a ridge $R$ **positively** if $\exists v \in V \cap R$ such that $(v - p) \cdot n > 0$, where $n$ is the outward pointing normal to $R$. 

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Claim: For every sequence $S = (\ldots, F)$, there is a frustum $\mathcal{V}(S)$ such that $\mathcal{E}(S) = \mathcal{V}(S) \cap F$. 
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To prove this (and also complete the algorithm to draw the visual field)

- We show this is true for the root of $\Gamma$
- We compute $\mathcal{V}(S')$, where $S'$ is a child of $S$ and $\mathcal{V}(S)$ is known.
Claim: For every sequence $S = (\ldots, F)$, there is a frustum $\mathcal{V}(S)$ such that $\mathcal{E}(S) = \mathcal{V}(S) \cap F$.

To prove this (and also complete the algorithm to draw the visual field)

- We show this is true for the root of $\Gamma$.
- We compute $\mathcal{V}(S')$, where $S'$ is a child of $S$ and $\mathcal{V}(S)$ is known.

The claim is trivial for the root $(F_0)$ of $\Gamma$, since $\mathcal{E}(F_0) = \mathbb{R}^n \cap F_0$. For the inductive step, we will do an example.
Visualization

The inductive step

Data for sequence $S = (\ldots, F)$

The frustum intersects ridges $\alpha$ and $\beta$ positively.
Visualization

The inductive step

Data for child node $S' = (\ldots, F, \alpha, A)$

$V(S') = (\text{red frustum}) \cap (\text{black frustum})$
Visualization

The inductive step

Data for child node $S'' = (\ldots, F, \beta, B)$

$\mathcal{V}(S'') = (\text{red frustum}) \cap (\text{black frustum})$
Visualization

The algorithm

Let $S = (\ldots, F)$ be a sequence for which $\mathcal{V}(S)$ is known.
Let $S' = (\ldots, F, R, F')$ be a child of $S$.
The following returns $\mathcal{V}(S')$:
Visualization
The algorithm

Let \( S = (\ldots, F) \) be a sequence for which \( \mathcal{V}(S) \) is known.
Let \( S' = (\ldots, F, R, F') \) be a child of \( S \).
The following returns \( \mathcal{V}(S') \):

Algorithm

```plaintext
if( \( \mathcal{V}(S) \) intersects \( R \) positively ){
    p = Source point of \( \mathcal{V}(S) \);
    \( V_R = \) Frustum generated from \( p \) and \( R \);
    return \( \mathcal{T}(F, R)(V_R \cap \mathcal{V}(S)) \);
}
else{ return \( \emptyset \); }
```

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